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Intrinsic complexity of learning geometrical concepts from positive data

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Abstract

Intrinsic complexity is used to measure the complexity of learning areas limited by broken-straight lines (called *open semi-hulls*) and intersections of such areas. Any strategy learning such geometrical concepts can be viewed as a sequence of *primitive basic* strategies. Thus, the length of such a sequence together with the complexities of the primitive strategies used can be regarded as the complexity of learning the concepts in question. We obtained the best possible lower and upper bounds on learning open semi-hulls, as well as matching upper and lower bounds on the complexity of learning intersections of such areas. Surprisingly, upper bounds in both cases turn out to be much lower than those provided by natural learning strategies. Another surprising result is that learning intersections of open semi-hulls turns out to be easier than learning open semi-hulls themselves.

1. Introduction

Learning geometrical concepts from examples is a popular topic in Computational Learning Theory (see for example, [BEHW89,BGGM98,BGM98,CA99,CM94,DG95,Fis95, GG94,GGDM94,GGS96,GKS01,GS99,Heg94]). The above-listed papers mostly dealt with finite geometric concepts. The goal of this paper is to quantify the complexity of algorithmic learning of *infinite* geometrical concepts from growing finite segments. For this purpose, we will be using the learning in the limit model.

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Consider, for example, an *open semi-hull* representing the space consisting of all points (x, y) with integer components x, y in the first quadrant of the plane bounded by the y-axis and the broken line passing through some points $(a_0, c_0), (a_1, c_1), \ldots, (a_n, c_n), a_i, c_i \in N, a_i < a_{i+1}, 0 \leqslant i < n$. The line is straight between any points $(a_i, c_i), (a_{i+1}, c_{i+1})$ and begins at $(a_0, c_0) = (0, 0)$; further we assume that the slope of the broken line is monotonically non-decreasing—that is $(c_{i+1} - c_i)/(a_{i+1} - a_i)$ is non-decreasing in i (we need this for learnability of the semi-hull). For technical ease we further assume that the first line segment $(0, 0), (a_1, c_1)$ is adjacent to the x-axis, that is, $c_1 = 0$. (See example semi-hull figure in Fig. 1.) Note that each break point in the boundary of the semi-hull defines an angle in the semi-hull. Any such open semi-hull can be easily learned in the limit by the following strategy: given growing finite sets of points in the open semi-hull (potentially getting all points), learn the first break point (a_1, c_1). Here, under our assumption, c_1 must be 0, therefore, we change our mind every time when we get in the input a new point $(a, 0)$ with the value a greater than all values b in the points $(b, 0)$ seen so far. Now, once the first break point has been learned, we can try to learn the first slope $(c_2 - c_1)/(a_2 - a_1)$. The more points we get in the input, the more our hypothetical border line may bend towards the x-axis. Since it can never cross the x-axis and since the points in the concept to be learned have integer components, we will eventually learn the slope. Now, moving along the border line as more points on it become available from the input, we can learn the second break point (a_2, c_2). Then we can learn the second slope $(c_3 - c_2)/(a_3 - a_2)$, etc. Is this strategy the best possible? Do there exist easier strategies of learning such geometrical concepts? How do we measure the

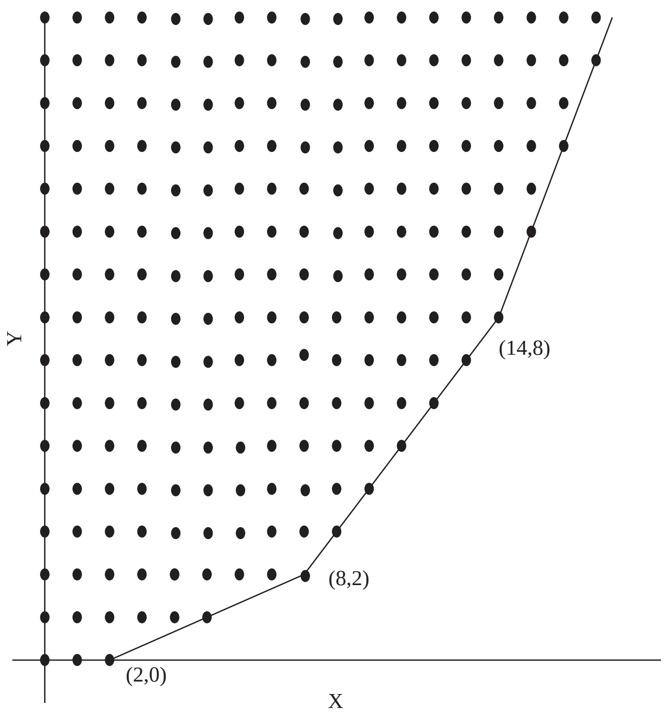

Fig. 1. Open semi-hull.

complexity of learning such concepts? As we are interested in learning infinite objects from infinitely growing finite segments, we use inductive inference as our learning paradigm. This paradigm suggests several ways to quantify the complexity of learning. Among them are:

(a) counting the number of *mind changes* [BF72,CS83,LZ93] the learner makes before arriving at the correct hypothesis;

(b) measuring the amount of so-called *long-term* memory the learner uses [KS95];

(c) reductions between different learning problems (classes of languages) and respective degrees of the so-called *intrinsic* complexity [FKS95,JS96,JS97].

There have been other notions of the complexity of learning in the limit considered in literature (for example see [DS86,Gol67,Wie86]).

The first two approaches, however, cannot capture the complexity of learning open semi-hulls with different numbers of angles: the number of mind changes cannot be bounded by any reasonable function (even for learning the very first break point), and the long-term memory is maximum (linear) even for one angle. Thus, we have chosen reductions as the way to measure the (relative) complexity of geometrical concepts like open semi-hulls. An important issue here is which classes of languages can be used as a *scale* for quantifying the complexity of open semi-hulls. One such scale, that turned out to be appropriate for our goal, had been suggested in [JKW99,JKW00]. This scale is a hierarchy of degrees of intrinsic complexity composed of simple natural ground degrees. Every such natural ground degree represents a natural type of learning strategy. For example, the degree *INIT* represents a strategy that tries to use a sequence of hypotheses equivalent to a sequence of monotonically growing finite sets. Another such strategy, *COINIT*, tries to use a sequence of hypotheses equivalent to a sequence of monotonically growing sets $N_a = \{x | x \in N, x \geqslant a\}$. Intuitively, capabilities of *INIT*- and *COINIT*-strategies must be different (this has been formally established in [JS96]). For example, when a *COINIT*-strategy learns the language $\{5, 6, 7, \ldots\}$, it actually tries to find the minimum number in this set (5 in our case). When the first number, say, 17 appears in the input, the strategy can immediately use it as the upper bound on its conjectures: no number greater than 17 can possibly be the desired minimum. Note also that the strategy is aware of the absolute lower bound 0 for all its conjectures. On the other hand, an *INIT*-strategy learning, say, the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ tries to find the maximum number 7. While, every next inputted number may increase the lower bound on its final conjecture, the strategy never is aware of any upper bound on it.

It has been demonstrated in [JS96] that many important simple learning problems (in particular, *pattern languages*) can be handled by strategies of these basic types. Now, the corresponding degrees of complexity *INIT* and *COINIT* can be used to form a complex hierarchy of degrees as follows. Imagine a three-dimensional language L. Suppose an *INIT*-type strategy M_1 can be used to learn its first dimension, L_1. Once this dimension has been learned, a strategy of a different (or even same) type, say, *COINIT* can pick the grammar learned by M_1 and use this information to learn the second dimension L_2. Consequently, the grammar learned by the *COINIT*-strategy M_2 can be used to learn the third dimension L_3 by a strategy M_3 of type, say, *INIT*. Thus, we get a strategy of the type $(INIT, COINIT, INIT)$, where information learned by the learner M_i is relayed to the learner M_{i+1} making it possible to learn the next dimension. This idea can be naturally extended to any (finite) number of dimensions and to any sequences $Q = (q_1, q_1, \ldots, q_k)$ of strategies $q_i \in \{INIT, COINIT\}$. It has been shown in [JKW00] that the degrees of complexity (classes of languages) corresponding to such Q-strategies form a rich hierarchy. For example, some classes learnable by $(INIT, COINIT, INIT)$-type strategies cannot be learned by any $(INIT, COINIT)$-strategy. In other words, such a class can be learned by a strategy that first works as an *INIT*-strategy, then it changes to a *COINIT*-strategy to learn another aspect of the concept, and then it changes to a *INIT*-strategy to learn the last component of the concept. On the other hand, changing strategies only one time (from *INIT* to *COINIT*) is not enough.

How can one apply the above hierarchy to quantify the complexity of learning semi-hulls? Let us take a closer look at the strategy learning semi-hulls (described in the beginning of the paper). The reader may have noticed that the strategy learning the first break point (as well as any other break point) is an *INIT*-type strategy: it tries to find the maximum number in a growing finite set. Now, once a break point has been found, what strategy can learn the slope? Let us assume that the slopes come from the set $N \cup \{1/n \mid n \in N\}$ (actually, sets of possible slopes in our model will be equivalent to this set). Suppose the actual slope to be learned is $\frac{1}{7}$. Suppose also that, based on some initial portion of the input, we conjectured the slope 5. That tells us that the actual slope cannot be greater than 5. Therefore, every following conjecture will be a number between 5 and $\frac{1}{7}$, getting closer and closer to $\frac{1}{7}$, but never being aware of any lower bound on its final guess $\frac{1}{7}$. This strategy seems to be similar to a *COINIT*-strategy discussed above. However, unlike a *COINIT*-strategy, it is never aware of any lower bound on possible conjectures. One can easily see that a very similar (equivalent) strategy can learn sets $\{\ldots -k, -(k-1), \ldots 0, 1, 2, 3, \ldots\}$. Based on this observation, we call such strategies *HALF*-strategies (learning "halfs" of the set of integers). We also show that these strategies, while being still quite primitive, are more powerful than *COINIT*-strategies (moreover, in certain sense, they can be regarded as Cartesian products of *INIT*- and *COINIT*-strategies).

Thus, to learn the first break point we use an *INIT*-strategy, then we use a *HALF*-strategy to learn the first slope, then we change back to an *INIT*-strategy to learn the second break point, then to a *HALF*-strategy to learn the second slope, etc. If, for example, a semi-hull has two angles (the border line has two break points), then we use the sequence of strategies *INIT*, *HALF*, *INIT*, *HALF*. In other words, the corresponding learning problem belongs to the level $(INIT, HALF, INIT, HALF)$ of the above-mentioned hierarchy. Obviously, learning the semi-hulls with two angles by an $(INIT, HALF, INIT)$-strategy (if possible) can be viewed as more efficient. We will show, however, that such a strategy is not possible. Another question is if semi-hulls with two angles are learnable by an $(INIT, COINIT, INIT, COINIT)$-strategy. One must again agree that this strategy can be considered as a more efficient than our $(INIT, HALF, INIT, HALF)$-strategy, since, as we mentioned above, *COINIT*-strategies generally are more primitive (less capable) than *HALF*-strategies. We will show that such a strategy does exist for semi-hulls with two angles (Theorem 3; it is somewhat less intuitive than our original $(INIT, HALF, INIT, HALF)$-strategy). We will also show that for example, neither $(COINIT, INIT, COINIT, INIT)$-strategy nor any strategy with a number of components (from $INIT, COINIT, HALF$) smaller than 4 can learn the given class, as it easily follows from Corollary 6. In general, for every class of semi-hulls with fixed numbers of angles (two, or three, or four, etc.) we establish upper and lower bounds of this type.

We submit that this approach to measuring the complexity of learning is very reasonable for geometrical concepts of this and similar types. As we already mentioned, some more traditional measures of complexity (like the number of mind changes) are not applicable, since these measures cannot make distinction between learning one-angle semi-hulls and, say, three-angle ones.

We use also a similar approach to examine the power of learning semi-hulls (and other figures) from a slightly different perspective. Namely, suppose we are given some strategy to learn two-angle semi-hulls. How can knowledge of such a strategy help us to learn problems like $(INIT, COINIT, INIT)$ (viewed this time as families of languages rather than strategies)? We can show that, being armed with such a strategy, one can learn, for example, the class of languages $(HALF, INIT)$ (Theorem 5). On the other hand, being able to learn two-angle semi-hulls does not help to learn the class $(INIT, INIT, INIT)$ (Corollary 7) or $(COINIT, INIT, INIT)$ (Corollary 8). It also does not help to learn any similar class with *COINIT* as the second component (Corollaries 9 and 10). In this respect, $(HALF, INIT)$ can be regarded as a lower bound on the power of learning two-angle semi-hulls.

The paper has the following structure. Section 2 introduces notations and preliminaries. In Section 3, we define the reductions and the degrees of complexity. In Section 4, we give formal definition of the Q-classes and degrees. This definition extends the definition of the Q-classes in [JKW00]: in addition to the classes *INIT* and *COINIT*, we use in vectors Q a new class of strategies/languages, *HALF*, that turns out to be different from *INIT* and *COINIT* and is useful for classifying geometrical concepts. In Section 5, we show that the Q-hierarchy can be appropriately extended to the class of learning strategies/languages involving *HALF*. In Section 6, we define the classes of the type *SEMI_HULL* that formalize intuitive geometrical concepts described above. In this section, we also prove some useful technical propositions. In Sections 7 and 8 we establish upper and lower bounds for the *SEMI_HULL* degrees in terms of the Q-hierarchy. In particular, we establish that semi-hulls with n angles can be learned by a $(INIT, COINIT, \ldots, INIT, COINIT)$-strategy with $2n$ components (Theorem 3). On the other hand, no strategy, with alternating *INIT*s and *COINIT*s and a smaller number of components, can learn such concepts (Corollary 6). We also examine how learning semi-hulls can help to learn the Q-classes (Theorem 5). Corollaries 7–10 give examples of the Q-classes that cannot be learned by strategies armed with a strategy capable of learning semi-hulls. These examples show that the Q-classes found in Theorem 5 are the most complex Q-classes that can be learned using a strategy for learning semi-hulls. Therefore, they can be regarded as the best possible lower bounds (based on the Q-classes considered here) on the power of learning semi-hulls.

In Section 9, we introduce the classes *coSEMI_HULL* that consist of complements of languages in *SEMI_HULL*. Sections 10 and 11 establish lower and upper bounds for *coSEMI_HULL*s in terms of the Q-hierarchy. In particular, we show that the most efficient strategy for learning complements of semi-hulls with n angles is $(INIT, COINIT, \ldots, COINIT)$ with n occurrences of *COINIT* (Theorem 10). We also give examples of the Q-classes that can and cannot be learned by strategies having access to strategies learning *coSEMI_HULL*s, providing the best possible lower bounds for the learnability power of complements of semi-hulls.

Upper and lower bounds for *SEMI_HULL*s and *coSEMI_HULL*s come close, but do not match (though, upper bounds are much lower than the ones suggested by intuitive strategies learning the classes in question). In Section 12, we define the classes of *open hulls* formed by intersections of languages in *SEMI_HULL*s adjacent to x- and y-axis; Fig. 2 shows an example of open hull. For the complexity of learning these classes, we have established matching upper and lower bounds in terms of the Q-hierarchy. Namely, we show that the most efficient strategy for

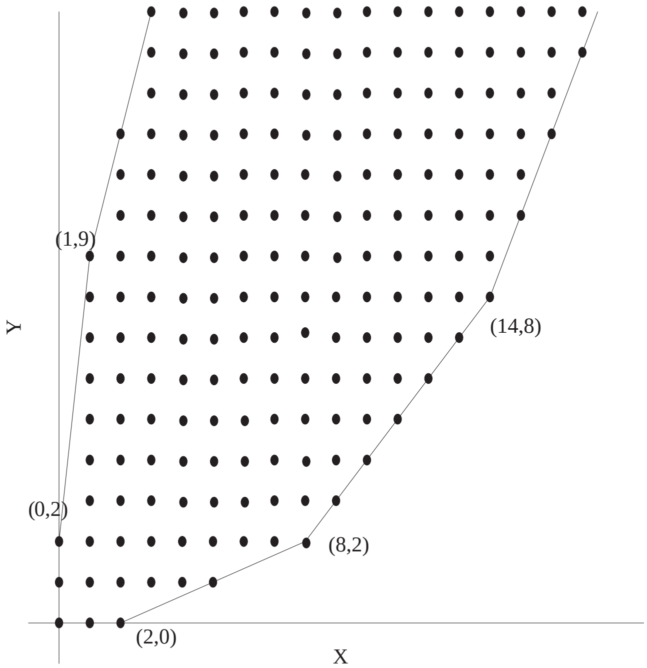

Fig. 2. Open hull.

learning open hulls with at most n angles on each side is $(INIT, INIT, \ldots, INIT)$ with n components $INIT$ (Theorem 14). It turns out also (Theorem 13) that a strategy armed with a strategy for learning open hulls can learn the class $(INIT, INIT, \ldots INIT)$ with n components. (It seems a bit counterintuitive that open hulls can be learned by strategies simpler than semi-hulls. We will give some intuition behind the corresponding strategy in Theorem 14).

In Section 13, we define the classes of languages formed by complements of open hulls and establish matching upper and lower bounds for the corresponding degrees of intrinsic complexity. All the above-mentioned upper bounds are much lower than the ones suggested by intuitive learning strategies.

2. Notation and preliminaries

Any unexplained recursion theoretic notation is from [Rog67]. The symbol N denotes the set of natural numbers, $\{0, 1, 2, 3, \ldots\}$. Z denotes the set of integers. Z^- denotes the set of negative integers. $i \dot{-} j$ is defined as follows:

$$i \dot{-} j = \begin{cases} i - j, & \text{if } i \geqslant j, \\ 0, & \text{otherwise.} \end{cases}$$

Symbols $\emptyset$, $\subseteq$, $\subset$, $\supseteq$, and $\supset$ denote empty set, subset, proper subset, superset, and proper superset, respectively. $D_0, D_1, \ldots,$ denotes a canonical recursive indexing of all the finite sets [Rog67]. We assume that if $D_i \subseteq D_j$ then $i \leqslant j$ (the canonical indexing defined in [Rog67] satisfies this property). Cardinality of a set S is denoted by $card(S)$. The maximum and minimum of a set are denoted by $\max(\cdot), \min(\cdot)$, respectively, where $\max(\emptyset) = 0$ and $\min(\emptyset) = \infty$.

We let $\langle \cdot, \cdot \rangle$ stand for an arbitrary, computable, bijective mapping from $N \times N$ onto N [Rog67]. We assume without loss of generality that $\langle \cdot, \cdot \rangle$ is monotonically increasing in both its arguments. We define $\pi_1(\langle x, y \rangle) = x$ and $\pi_2(\langle x, y \rangle) = y$. $\langle \cdot, \cdot \rangle$ can be extended to n-tuples in a natural way (including $n = 1$, where $\langle x \rangle$ may be taken to be x). Projection functions $\pi_1, \ldots, \pi_n$ corresponding to n-tuples can be defined similarly (where the tuple size would be clear from context). Due to above isomorphism between N^n and N, we often identify $(x_1, \ldots, x_n)$ with $\langle x_1, \ldots, x_n \rangle$. Thus we can say $L_1 \times L_2 = \{\langle x, y \rangle \mid x \in L_1, y \in L_2\}$.

By φ we denote a fixed *acceptable* programming system for the partial computable functions: $N \to N$ [Rog67,MY78]. By φ_i we denote the partial computable function computed by the program with number i in the φ-system. Symbol $\mathscr{R}$ denotes the set of all total computable functions. By Φ we denote an arbitrary fixed Blum complexity measure [Blu67,HU79] for the φ-system. By W_i we denote domain(φ_i). W_i is, then, the r.e. set/language ($\subseteq N$) accepted (or equivalently, generated) by the φ-program i. We also say that i is a grammar for W_i. Symbol $\mathscr{E}$ will denote the set of all r.e. languages. Symbol L, with or without decorations, ranges over $\mathscr{E}$. By $\bar{L}$, we denote the complement of L, that is $N - L$. Symbol $\mathscr{L}$, with or without decorations, ranges over subsets of $\mathscr{E}$. We denote by $W_{i,s}$ the set $\{x < s \mid \Phi_i(x) < s\}$.

$\downarrow$ denotes defined or converges. $\uparrow$ denotes undefined or diverges.

A partial function F from N to N is said to be partial limit recursive, iff there exists a recursive function f from $N \times N$ to N such that for all x, $F(x) = \lim_{y \to \infty} f(x, y)$. Here if $F(x)$ is not defined then $\lim_{y \to \infty} f(x, y)$, must also be undefined. A partial limit recursive function F is called (total) limit recursive function, if F is total.

We now present concepts from language learning theory. The next definition introduces the notion of a *sequence* of data.

Definition 1. (a) A *finite sequence* σ is a mapping from an initial segment of N into $(N \cup \{\#\})$. The empty sequence is denoted by Λ.

(b) The *content* of a finite sequence σ, denoted *content*(σ), is the set of natural numbers in the range of σ.

(c) The *length* of σ, denoted by $|\sigma|$, is the number of elements in σ. So, $|\Lambda| = 0$.

(d) For $n \leqslant |\sigma|$, the initial sequence of σ of length n is denoted by $\sigma[n]$. So, $\sigma[0]$ is Λ.

Intuitively, #'s represent pauses in the presentation of data. We let σ, τ, and γ, with or without decorations, range over finite sequences. We denote the sequence formed by the concatenation of τ at the end of σ by $\sigma \diamond \tau$. Sometimes we abuse the notation and use $\sigma \diamond x$ to denote the concatenation of sequence σ and the sequence of length 1 which contains the element x. *SEQ* denotes the set of all finite sequences.

Definition 2 (Gold [Gol67]). (a) A *text* T for a language L is a mapping from N into $(N \cup \{\#\})$ such that L is the set of natural numbers in the range of T.

(b) The *content* of a text T, denoted *content*(T), is the set of natural numbers in the range of T.

(c) $T[n]$ denotes the finite initial sequence of T with length n.

We let T, with or without decorations, range over texts. We let $\mathscr{T}$ range over sets of texts.

Definition 3 (Gold [Gol67]). A *language learning machine* is an algorithmic device which computes a mapping from SEQ into N.

We let $\mathbf{M}$, with or without decorations, range over learning machines. $\mathbf{M}(T[n])$ is interpreted as the grammar (index for an accepting program) conjectured by the learning machine $\mathbf{M}$ on the initial sequence $T[n]$. We say that $\mathbf{M}$ converges on T to i (written $\mathbf{M}(T)\downarrow = i$) iff $(\overset{\infty}{\forall} n)[\mathbf{M}(T[n]) = i]$.

There are several criteria for a learning machine to be successful on a language. Below we define identification in the limit introduced by Gold [Gol67].

Definition 4 (Gold [Gol67], Case and Smith [CS83]). Suppose $a \in N \cup \{*\}$.

(a) $\mathbf{M}$ $\mathbf{TxtEx}$-identifies a text T if and only if $(\exists i \mid W_i = content(T))(\overset{\infty}{\forall} n)[\mathbf{M}(T[n]) = i]$.

(b) $\mathbf{M}$ $\mathbf{TxtEx}$-identifies an r.e. language L (written: $L \in \mathbf{TxtEx}(\mathbf{M})$) if and only if $\mathbf{M}$ $\mathbf{TxtEx}$-identifies each text for L.

(c) $\mathbf{M}$ $\mathbf{TxtEx}$-identifies a class $\mathscr{L}$ of r.e. languages (written: $\mathscr{L} \subseteq \mathbf{TxtEx}(\mathbf{M})$) iff $\mathbf{M}$ $\mathbf{TxtEx}$-identifies each $L \in \mathscr{L}$.

(d) $\mathbf{TxtEx} = \{\mathscr{L} \subseteq \mathscr{E} \mid (\exists \mathbf{M})[\mathscr{L} \subseteq \mathbf{TxtEx}(\mathbf{M})]\}$.

Other criteria of success are finite identification [Gol67], behaviorally correct identification [CL82,Fel72,OW82], and vacillatory identification [Cas99,OW82]. In the present paper, we only discuss results about $\mathbf{TxtEx}$-identification.

3. Reductions

We first present some technical machinery.

We write $\sigma \subseteq \tau$ if σ is an initial segment of τ, and $\sigma \subset \tau$ if σ is a proper initial segment of τ. Likewise, we write $\sigma \subset T$ if σ is an initial finite sequence of text T. Let finite sequences $\sigma^0, \sigma^1, \sigma^2, \ldots$ be given such that $\sigma^0 \subseteq \sigma^1 \subseteq \sigma^2 \subseteq \cdots$ and $\lim_{i\to\infty} |\sigma^i| = \infty$. Then there is a unique text T such that for all $n \in N$, $\sigma^n = T[|\sigma^n|]$. This text is denoted by $\bigcup_n \sigma^n$. Let $\mathbf{T}$ denote the set of all texts, that is, the set of all infinite sequences over $N \cup \{\#\}$.

We define an *enumeration operator* (or just operator), Θ, to be an algorithmic mapping from SEQ into SEQ such that for all $\sigma, \tau \in SEQ$, if $\sigma \subseteq \tau$, then $\Theta(\sigma) \subseteq \Theta(\tau)$. We further assume that for all texts T, $\lim_{n\to\infty} |\Theta(T[n])| = \infty$. By extension, we think of Θ as also defining a mapping from $\mathbf{T}$ into $\mathbf{T}$ such that $\Theta(T) = \bigcup_n \Theta(T[n])$.

A final notation about the operator Θ: If for a language L, there exists an L' such that for each text T for L, $\Theta(T)$ is a text for L', then we write $\Theta(L) = L'$, else we say that $\Theta(L)$ is undefined. The reader should note the overloading of this notation because the type of the argument to Θ could be a sequence, a text, or a language; it will be clear from the context which usage is intended.

We let $\Theta(\mathscr{T}) = \{\Theta(T) \mid T \in \mathscr{T}\}$, and $\Theta(\mathscr{L}) = \{\Theta(L) \mid L \in \mathscr{L}\}$.

We also need the notion of an infinite sequence of grammars. We let α, with or without decorations, range over infinite sequences of grammars. From the discussion in the previous section, it is clear that infinite sequences of grammars are essentially infinite sequences over N. Hence, we adopt the machinery defined for sequences and texts over to finite sequences of grammars and infinite sequences of grammars. So, if $\alpha = i_0, i_1, i_2, i_3, \ldots$, then $\alpha[3]$ denotes the sequence i_0, i_1, i_2, and $\alpha(3)$ is i_3. Furthermore, we say that α converges to i if there exists an n such that, for all $n' \geqslant n$, $i_{n'} = i$.

Let **I** be any criterion for language identification from texts, for example $\mathbf{I} = \mathbf{TxtEx}$. We say that an infinite sequence α of grammars is **I**-*admissible* for text T just in case α witnesses **I**-identification of text T. So, if $\alpha = i_0, i_1, i_2, \ldots$ is a **TxtEx**-admissible sequence for T, then α converges to some i such that $W_i = \mathit{content}(T)$; that is, the limit i of the sequence α is a grammar for the language $\mathit{content}(T)$.

Intuitively, a learning problem $\mathscr{L}_1$ is reducible to a learning problem $\mathscr{L}_2$ if, using a strategy learning $\mathscr{L}_1$ one can learn $\mathscr{L}_2$.

We now formally introduce our reductions. Although in this paper we will only be concerned with **TxtEx**-identification, we present the general case of the definition.

Definition 5 (Jain and Sharma [JS96]). Let $\mathscr{L}_1 \subseteq \mathscr{E}$ and $\mathscr{L}_2 \subseteq \mathscr{E}$ be given. Let identification criterion **I** be given. Let $\mathscr{T}_1 = \{T \mid T \text{ is a text for } L \in \mathscr{L}_1\}$. Let $\mathscr{T}_2 = \{T \mid T \text{ is a text for } L \in \mathscr{L}_2\}$. We say that $\mathscr{L}_1 \leqslant^{\mathbf{I}} \mathscr{L}_2$ if and only if there exist operators Θ and Ψ such that, for all $T, T' \in \mathscr{T}_1$ and for all infinite sequences α of grammars, the following hold:

(a) $\Theta(T) \in \mathscr{T}_2$ and
(b) if α is an **I**-admissible sequence for $\Theta(T)$, then $\Psi(\alpha)$ is an **I**-admissible sequence for T.
(c) if $\mathit{content}(T) = \mathit{content}(T')$ then $\mathit{content}(\Theta(T)) = \mathit{content}(\Theta(T'))$.

We say that $\mathscr{L}_1 \equiv^{\mathbf{I}} \mathscr{L}_2$ iff $\mathscr{L}_1 \leqslant^{\mathbf{I}} \mathscr{L}_2$ and $\mathscr{L}_2 \leqslant^{\mathbf{I}} \mathscr{L}_1$.

The reduction defined above was called strong-reduction in [JS96]. The above reduction without clause (c) was called weak reduction. Since in this paper we will be only concerned with strong reductions, we just refer to them as reductions. (Weak reductions are not sharp enough to provide real distinction between most of the classes considered in this paper).

Intuitively, $\mathscr{L}_1 \leqslant^{\mathbf{I}} \mathscr{L}_2$ just in case there exists an operator Θ that transforms texts for languages in $\mathscr{L}_1$ into texts for languages in $\mathscr{L}_2$ and there exists another operator Ψ that behaves as follows: if Θ transforms text T (for a language in $\mathscr{L}_1$) to text T' (for a language in $\mathscr{L}_2$), then Ψ transforms **I**-admissible sequences for T' into **I**-admissible sequences for T. Thus, informally, the operator Ψ has to work only on **I**-admissible sequences for such texts T'. In other words, if α is a sequence of grammars which is *not* **I**-admissible for any text T' in $\{\Theta(T) \mid \mathit{content}(T) \in \mathscr{L}_1\}$, then $\Psi(\alpha)$ can be defined *arbitrarily*. This property will be used implicitly at all places below where we have to define operators Ψ witnessing (together with operators Θ) some reducibility. Note that this approach both simplifies the corresponding definitions and preserves the computability of the so-defined operators.

Additionally different texts for some language $L \in \mathscr{L}_1$, are transformed into (possibly different) texts for same language $L' \in \mathscr{L}_2$.

Now, a *degree* of learnability under our reduction is, naturally, a set of families $\mathscr{L}$ reducible to each other (i.e. the equivalence class under the reduction considered).

Intuitively, for many identification criteria **I** such as **TxtEx**, if $\mathscr{L}_1 \leqslant^{\mathbf{I}} \mathscr{L}_2$ then the problem of identifying $\mathscr{L}_2$ in the sense of **I** is at least as hard as the problem of identifying $\mathscr{L}_1$ in the sense of **I**, since the solvability of the former problem implies the solvability of the latter one. That is given any machine $\mathbf{M}_2$ which **I**-identifies $\mathscr{L}_2$, one can construct a machine $\mathbf{M}_1$ which **I**-identifies $\mathscr{L}_1$. To see this, for $\mathbf{I} = \mathbf{TxtEx}$, suppose Θ and Ψ witness $\mathscr{L}_1 \leqslant^{\mathbf{I}} \mathscr{L}_2$. $\mathbf{M}_1(T)$, for a text T is defined as follows. Let $p_n = \mathbf{M}_2(\Theta(T)[n])$, and $\alpha = p_0, p_1, \ldots$. Let $\alpha' = \Psi(\alpha) = p'_0, p'_1, \ldots$. Then let $\mathbf{M}_1(T) = \lim_{n \to \infty} p'_n$. Consequently, $\mathscr{L}_2$ may be considered as a hardest problem for **I**-identification if for *all* classes $\mathscr{L}_1 \in \mathbf{I}$, $\mathscr{L}_1 \leqslant^{\mathbf{I}} \mathscr{L}_2$ holds. If $\mathscr{L}_2$ itself belongs to **I**, then $\mathscr{L}_2$ is said to be *complete*. We now formally define these notions of hardness and completeness for the above reduction.

Definition 6 (Jain and Sharma [JS96]). Let **I** be an identification criterion. Let $\mathscr{L} \subseteq \mathscr{E}$ be given.

(a) If for all $\mathscr{L}' \in \mathbf{I}$, $\mathscr{L}' \leqslant^{\mathbf{I}} \mathscr{L}$, then $\mathscr{L}$ is $\leqslant^{\mathbf{I}}$-*hard*.
(b) If $\mathscr{L}$ is $\leqslant^{\mathbf{I}}$-hard and $\mathscr{L} \in \mathbf{I}$, then $\mathscr{L}$ is $\leqslant^{\mathbf{I}}$-*complete*.

Proposition 1 (Jain and Sharma [JS96]). $\leqslant^{\mathbf{TxtEx}}$ *is reflexive and transitive.*

The above proposition holds for most natural learning criteria.

Proposition 2 (Based on [JS97]). *Suppose* $\mathscr{L} \leqslant^{\mathbf{I}} \mathscr{L}'$, *via* Θ *and* Ψ. *Then, for all* $L, L' \in \mathscr{L}$, $L \subseteq L' \Rightarrow \Theta(L) \subseteq \Theta(L')$.

We will be using Proposition 2 implicitly when we are dealing with reductions. Since, for $\mathscr{L} \leqslant^{\mathbf{I}} \mathscr{L}'$ via Θ and Ψ, for all $L \in \mathscr{L}$, $\Theta(L)$ is defined (= some $L' \in \mathscr{L}'$), when considering reductions, we often consider Θ as mapping finite sets to (possibly infinite) sets instead of mapping sequences to sequences.[1] This is clearly without loss of generality, as one can easily convert such Θ to Θ as in Definition 5 of reduction.

4. Q-classes

In this section, we introduce the classes of languages and corresponding degrees of intrinsic complexity that form the scale being used for estimating the complexity of learning open semi-hulls and open hulls. First, we define ground natural classes that are being used as bricks to build our hierarchy of degrees.

[1] For infinite X, by definition, $\Theta(X)$ would be $\bigcup_{X' \subseteq X \,:\, card(X') < \infty} \Theta(X')$.

Definition 7. $INIT = \{L \subseteq N \mid (\exists i \in N)[L = \{x \in N \mid x \leqslant i\}]\}$.
$COINIT = \{L \subseteq N \mid (\exists i \in N)[L = \{x \in N \mid x \geqslant i\}]\}$.
$HALF = \{L \subseteq Z \mid (\exists i \in Z)[L = \{x \in Z \mid x \geqslant i\}]\}$.

Note that officially our definition for languages and r.e. sets as in Section 2, only allows subsets of N. Since, one can easily code Z onto N, by slight abuse of convention, we can consider subsets of Z also as languages. We thus assume an implicit coding of Z onto N whenever we deal with languages and language classes involving *HALF*, without explicitly stating so.

In the sequel, we will use the above notation in two different contexts. Namely, we will use *INIT* to denote the class of languages as defined above, and to denote the degree of *all* classes of languages equivalent to the class *INIT* under our reductions. Similarly, we will use in two different contexts *COINIT*, *HALF* and all their combinations defined below in this section. In every use of this notation, the reader will be able to easily determine an appropriate context.

While both the classes *INIT* and *COINIT* are monotonically learnable, the types of conjectures being used to learn *INIT* and, respectively, *COINIT* are obviously different. For *INIT* the maximum element in the input gives a code for the language, whereas in *COINIT* the minimum element gives the code. Note that the maximum element used in *INIT* strategy is unbounded, whereas the minimum element for *COINIT* is bounded by 0. So, not surprisingly, the degrees of *INIT* and *COINIT* were proven in [JS96] to be different.

Classes *HALF* and *COINIT* are learnable by similar strategies, however, the minimum element in *HALF* is unbounded. We will formally prove below that degree of *HALF* is different from degrees of both *INIT* and *COINIT*. Furthermore, we will show that the degree of *HALF* can be viewed as a cross product of the degrees of *INIT* and *COINIT*.

There are several other natural classes considered in the literature such as *FINITE* (degree of which is equivalent to *INIT*), *SINGLE*, *COSINGLE*, etc. but we will not be concerned with them here since they will not be relevant to our results.

Now we define the cross product of arbitrary classes $\mathscr{L}_1$ and $\mathscr{L}_2$.

Definition 8. Let $\mathscr{L}_1, \mathscr{L}_2$ be two classes of languages. Then $\mathscr{L}_1 \times \mathscr{L}_2 = \{L_1 \times L_2 \mid L_1 \in \mathscr{L}_1, L_2 \in \mathscr{L}_2\}$.

This definition can be naturally extended to any finite number of dimensions. For example, one can naturally define $\mathscr{L}_1 \times \mathscr{L}_2 \times \mathscr{L}_3$, etc.

Theorem 1. $HALF \equiv^{\mathbf{TxtEx}} INIT \times COINIT$.

Proof of the above theorem is given in Appendix A.

Now, following [JKW00], we are going to combine the classes *INIT*, *COINIT*, and *HALF* to form classes of multidimensional languages, where, to learn the dimension L_{k+1} of a language L, the learner must first learn the parameters $i_1, \ldots, i_k$ of the dimensions $L_1, \ldots, L_k$; then L_{k+1} is the projection $\{x_{k+1} \mid \langle i_1, \ldots, i_k, x_{k+1}, x_{k+2}, \ldots, x_n \rangle \in L\}$ with a simple sublanguage whose description is specified yet by $i_1, \ldots, i_k$. Once it has been determined which projection must be learned, the learner can use a predefined *INIT*-, *COINIT*-, or *HALF*-type strategy to learn the projection in question. For example, one can consider a class of two-dimensional languages $(INIT, COINIT)$, where the first dimension $L_1 = \{x | \langle x, y \rangle \in L\}$ of any language L belongs to $INIT$, and if i is the parameter describing L_1 (that can be learned by an $INIT$-type strategy) then the projection $\{y | \langle i, y \rangle \in L\}$ is in $COINIT$.

Below for any tuples X and Y, let $X \cdot Y$ denote the concatenation of X and Y. That is if $X = \langle x_1, x_2, \ldots, x_n \rangle$ and $Y = \langle y_1, y_2, \ldots, y_m \rangle$ then $X \cdot Y = \langle x_1, \ldots, x_n, y_1, \ldots y_m \rangle$. Let $BASIC = \{INIT, COINIT, HALF\}$.

In part (c) of the following definition and in later situations in languages involving $HALF$ we sometimes abuse notation slightly and allow elements of Z as components of the pairing function $\langle \cdots \rangle$. This is for ease of notation, and one could easily replace these by using some coding of Z onto N.

Definition 9 (Jain et al. [JKW00]). Suppose $k \geqslant 1$. Let $Q \in BASIC^k$. Let $I \in N^k$. Then inductively on k, we define the languages L_I^Q and $T(L_I^Q)$ and $P(L_I^Q)$ as follows.

If $k = 1$, then

(a) if $Q = (INIT)$ and $I = (i)$, $i \in N$, then

$$T(L_I^Q) = \{\langle x \rangle \mid x \in N, x < i\}, \; P(L_I^Q) = \{\langle i \rangle\} \quad \text{and} \quad L_I^Q = T(L_I^Q) \cup P(L_I^Q).$$

(b) if $Q = (COINIT)$ and $I = (i)$, $i \in N$, then

$$T(L_I^Q) = \{\langle x \rangle \mid x \in N, x > i\}, \; P(L_I^Q) = \{\langle i \rangle\} \quad \text{and} \quad L_I^Q = T(L_i^Q) \cup P(L_i^Q).$$

(c) if $Q = (HALF)$ and $I = (i)$, $i \in Z$, then

$$T(L_I^Q) = \{\langle x \rangle \mid x \in Z, x > i\}, \; P(L_I^Q) = \{\langle i \rangle\} \quad \text{and} \quad L_I^Q = T(L_i^Q) \cup P(L_i^Q).$$

Now suppose we have already defined L_I^Q for $k \leqslant n$. We then define L_I^Q for $k = n + 1$ as follows. Suppose $Q = (q_1, \ldots, q_{n+1})$ and $I = (i_1, \ldots, i_{n+1})$. Let $Q_1 = (q_1)$ and $Q_2 = (q_2, \ldots, q_{n+1})$. Let $I_1 = (i_1)$ and $I_2 = (i_2, \ldots, i_{n+1})$. Then,

$$T(L_I^Q) = \{X \cdot Y \mid X \in T(L_{I_1}^{Q_1}), \text{ or } [X \in P(L_{I_1}^{Q_1}) \quad \text{and} \quad Y \in T(L_{I_2}^{Q_2})]\},$$

$$P(L_I^Q) = \{X \cdot Y \mid X \in P(L_{I_1}^{Q_1}) \quad \text{and} \quad Y \in P(L_{I_2}^{Q_2})\},$$

and

$$L_I^Q = T(L_I^Q) \cup P(L_I^Q).$$

Intuitively, in the above definition $T(L_I^Q)$ denotes the terminating part of the language that is specified yet by $i_1, \ldots, i_n, i_{n+1}$, and $P(L_I^Q)$ denotes the propagating part of the language L_I^Q that could be used for adding a language in dimension $n + 2$. (See [JKW00] for more details and motivation on the terminology of terminating and propagating.)

For ease of notation we often write $L_{(i_1, i_2, \ldots, i_k)}^Q$ as $L_{i_1, i_2, \ldots, i_k}^Q$.

Definition 10. Let $k \geqslant 1$. Let $Q = (q_1, \ldots, q_k) \in BASIC^k$ and $R = R_1 \times R_2 \times \cdots \times R_k$, where for $1 \leqslant i \leqslant k$, $R_i \subseteq N$ if $q_i \in \{INIT, COINIT\}$, and $R_i \subseteq Z$, if $q_i = HALF$. Then the class $\mathscr{L}^{Q,R}$ is defined as

$$\mathscr{L}^{Q,R} = \{L_I^Q \mid I \in R\}.$$

For technical convenience, for $Q = (\,)$, $I = (\,)$, $R = \{I\}$, we also define $T(L_I^Q) = \emptyset$, $P(L_Q^I) = \{\langle\ \rangle\}$, and $L_I^Q = T(L_I^Q) \cup P(L_I^Q)$, and $\mathscr{L}^{Q,R} = \{L_I^Q\}$.

Note that we have used a slightly different notation for defining the classes $\mathscr{L}^{Q,R}$ (for example instead of $INIT$, we now use $\mathscr{L}^{(INIT),N)}$). This is for clarity of notation.

Also, our main interest is for R_i's being N or Z (based on whether $q_i \in \{INIT, COINIT\}$ or $q_i = HALF$), though (as the following proposition shows) it does not matter as long as R_i is (or contains) an infinite recursive subset of N, if $q_i \in \{INIT, COINIT\}$, and R_i is (or contains) an infinite recursive subset of Z with infinite intersection with both N and Z^-, if $q_i = HALF$. The usage of general R is more for ease of proving some of our theorems.

Proposition 3. *Suppose $k \geqslant 1$. Let $Q \in BASIC^k$. Let $R = R_1 \times R_2 \times \cdots \times R_k$, where each R_i is an infinite recursive subset of N, if $q_i \in \{INIT, COINIT\}$ and R_i is a recursive subset of Z, with infinite intersection with both N and Z^-, if $q_i = HALF$. Let $R' = R'_1 \times R'_2 \times \cdots \times R'_k$, where R'_i is N, if $q_i \in \{INIT, COINIT\}$ and R'_i is Z, if $q_i = HALF$. Then, $\mathscr{L}^{Q,R} \equiv^{\mathbf{TxtEx}} \mathscr{L}^{Q,R'}$.*

For ease of notation, if $Q = (q_1, q_2, \ldots, q_n)$ and $R = R_1 \times R_2 \times \cdots \times R_n$, where $R_i = N$ if $q_i \in \{INIT, COINIT\}$ and $R_i = Z$ if $q_i = HALF$, then we drop R from $\mathscr{L}^{Q,R}$, using just $\mathscr{L}^Q$.

The immediate question is which classes Q represent different degrees.

Proposition 4. *Suppose $Q = (q_1, \ldots, q_{k-1}, q_k, q_{k+1}, \ldots, q_l)$ and $Q' = (q_1, \ldots, q_{k-1}, q', q_{k+1}, \ldots, q_l)$, where $q_k \in \{INIT, COINIT\}$ and $q' = HALF$, and other $q_i \in BASIC$.*

Then, $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$.

Proof. The above can be easily shown using $INIT \leqslant^{\mathbf{TxtEx}} HALF$ and $COINIT \leqslant^{\mathbf{TxtEx}} HALF$. $\square$

Since $INIT \times COINIT \leqslant^{\mathbf{TxtEx}} \mathscr{L}^Q$, for $Q = (INIT, COINIT)$, or $Q = (COINIT, INIT)$ [JKW00], we have the following proposition.

Proposition 5. *Suppose $Q = (q_1, \ldots, q_{k-1}, HALF, q_{k+1}, \ldots, q_n)$ and $Q' = (q_1, \ldots, q_{k-1}, INIT, COINIT, q_{k+1}, \ldots, q_n)$ or $Q' = (q_1, \ldots, q_{k-1}, COINIT, INIT, q_{k+1}, \ldots, q_n)$. Then $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$.*

Proof. The above can be easily shown using $HALF \leqslant^{\mathbf{TxtEx}} (INIT, COINIT)$ and $HALF \leqslant^{\mathbf{TxtEx}} (COINIT, INIT)$. $\square$

Proposition 6 (Based on [JS96]). *Suppose $Q = (INIT)$, $R = R_1$, $Q' = (COINIT)$, and $R' = R'_1$, where R_1 and R'_1 are infinite subsets of N. Then $\mathscr{L}^{Q,R} \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q',R'}$, and $\mathscr{L}^{Q',R'} \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q,R}$.*

The following technical definition introduces an ordering on all k-tuples of parameters $i_1, \ldots, i_k$ of languages in an arbitrary class $\mathcal{L}^Q$. This ordering will be helpful for the result in the next section.

Definition 11. Suppose $Q = (q_1, \ldots, q_k)$, where each $q_i \in \{HALF, INIT, COINIT\}$, for $1 \leqslant i \leqslant k$. Let $Q' = (q_2, \ldots, q_k)$. We say that $\langle i_1, \ldots, i_k \rangle \leqslant_Q \langle j_1, \ldots, j_k \rangle$ iff

(a) if $q_1 = INIT$, then $[i_1 < j_1]$ or $[i_1 = j_1$ and $\langle i_2, \ldots, i_k \rangle \leqslant_{Q'} \langle j_2, \ldots, j_k \rangle]$;
(b) if $q_1 = COINIT$ or $q_1 = HALF$, then $[i_1 > j_1]$ or $[i_1 = j_1$ and $\langle i_2, \ldots, i_k \rangle \leqslant_{Q'} \langle j_2, \ldots, j_k \rangle]$.

Here, for $Q = ()$, we assume that $\langle\,\rangle \leqslant_Q \langle\,\rangle$.

5. Q-hierarchy involving $HALF$

In this section, we establish a hierarchy among the Q-classes that will serve as a scale for the complexity of our geometrical concepts.

Definition 12. Q is said to be a pseudo-subsequence of Q', iff there exists a subsequence Q'' of Q', such that Q'' is obtainable from Q by

(i) replacing some $INIT$ with $HALF$,
(ii) replacing some $COINIT$ with $HALF$,
(iii) replacing some $HALF$ with $(COINIT, INIT)$, or
(iv) replacing some $HALF$ with $(INIT, COINIT)$.

For example, $Q = (INIT, HALF, COINIT)$ is a pseudo-subsequence of $Q' = (COINIT, HALF, COINIT, INIT, INIT, COINIT)$: dropping the first $COINIT$ and the second $INIT$ from Q', we get the subsequence $Q'' = (HALF, COINIT, INIT, COINIT)$; now Q can be promoted to Q'' by replacing $INIT$ with a more powerful $HALF$ and replacing $HALF$ (in its middle) with a more powerful combination $COINIT, INIT$.

In other words, a pseudo-subsequence of a vector Q is a subsequence where some $HALF$s are replaced with more primitive $INIT$s or $COINIT$s and some combinations $INIT, COINIT$ or $COINIT, INIT$ are replaced with more primitive $HALF$s.

Proposition 7. *Suppose $Q, Q', Q'' \in BASIC^*$. Then if Q is a pseudo-subsequence of Q' and Q' is a pseudo-subsequence of Q'', then Q is a pseudo-subsequence of Q''.*

Proof. Follows from definition of pseudo-subsequence. □

Proposition 8. *Suppose $Q, Q' \in BASIC^*$ and Q is a pseudo-subsequence of Q'. Then $\mathcal{L}^Q \leqslant^{\mathbf{TxtEx}} \mathcal{L}^{Q'}$.*

Proof. Follows from definition of pseudo-subsequence, Propositions 4 and 5. □

Proposition 9. *Suppose* $Q = (q_1, q_2, \ldots, q_k)$ *and* $Q' = (q'_1, q'_2, \ldots, q'_l)$, *where each* $q_i, q'_i \in BASIC$.

(a) *If* $q_1 = q'_1$, *then* $(q_2, \ldots, q_k)$ *is a pseudo-subsequence of* $(q'_2, \ldots, q'_l)$, *iff* Q *is a pseudo-subsequence of* Q'.

(b) *Suppose* $q_1, q'_1 \in \{INIT, COINIT\}$, *but* $q_1 \neq q'_1$. *Then,* Q *is a pseudo-subsequence of* Q' *iff* Q *is a pseudo-subsequence of* $Q'' = (q'_2, q'_3, \ldots, q'_l)$.

(c) *Suppose* $q_1 = HALF$. *Then,* Q *is a pseudo-subsequence of* Q' *implies* $(INIT, q_2, \ldots, q_k)$ *is a pseudo-subsequence of* Q'.

(d) *Suppose* $q_1 = HALF$. *Then,* Q *is a pseudo-subsequence of* Q' *implies* $(COINIT, q_2, \ldots, q_k)$ *is a pseudo-subsequence of* Q'.

(e) *Suppose* $q'_1 = HALF$. *Then,* Q *is a pseudo-subsequence of* Q' *iff* $(q_2, q_3, \ldots, q_k)$ *is a pseudo-subsequence of* $(q'_2, q'_3, \ldots, q'_l)$.

Proof. Follows from the definition of pseudo-subsequence. □

Proposition 10. *Suppose* $Q = (q_1, q_2, \ldots, q_k)$ *and* $Q' = (q'_1, q'_2, \ldots, q'_l)$, *where each* $q_i, q'_i \in BASIC$. *Suppose* Q *is not a pseudo-subsequence of* Q'.

(a) *If* $q_1 = q'_1 \in \{INIT, COINIT\}$, $q_2 \in \{INIT, COINIT\}$, *and* $q_2 \neq q_1$, *then* $Q'' = (q_2, q_3, \ldots, q_k)$ *is not a pseudo-subsequence of* Q'.

(b) *If* $q_1 = HALF$, *and* $q'_1 = INIT$, *then* $Q'' = (COINIT, q_2, \ldots, q_k)$ *is not a pseudo-subsequence of* Q'.

(c) *If* $q_1 = HALF$, *and* $q'_1 = COINIT$, *then* $Q'' = (INIT, q_2, \ldots, q_k)$ *is not a pseudo-subsequence of* Q'.

(d) *If* $q_1 = q'_1 \in \{INIT, COINIT\}$, $q_2 = HALF$, $l \geqslant 2$, *and* $q'_2 \in \{INIT, HALF\}$, *then* $Q'' = (q_1, COINIT, q_3, q_4, \ldots, q_k)$ *is not a pseudo-subsequence of* Q'.

(e) *If* $q_1 = q'_1 \in \{INIT, COINIT\}$, $q_2 = HALF$, $l \geqslant 2$, *and* $q'_2 \in \{COINIT, HALF\}$, *then* $Q'' = (q_1, INIT, q_3, q_4, \ldots, q_k)$ *is not a pseudo-subsequence of* Q'.

Proof. We show parts (a), (b) and (d). Proof of part (c) is similar to part (b) and proof of part (e) is similar to part (d).

(a) Let $Q''' = (q'_2, q'_3, \ldots, q'_l)$. Now by Proposition 9(a), Q is a pseudo-subsequence of Q' iff Q'' is pseudo-subsequence of Q'''. By Proposition 9(b), Q'' is pseudo-subsequence of Q''' iff Q'' is pseudo-subsequence of Q'. Thus, since Q is not a pseudo-subsequence of Q' it follows that Q'' is not a pseudo-subsequence of Q'.

(b) Suppose by way of contradiction that $(COINIT, q_2, \ldots, q_k)$ is a pseudo-subsequence of Q'. Then by applying Proposition 9(b), we have that $(COINIT, q_2, \ldots, q_k)$ is a pseudo-subsequence of $(q'_2, q'_3, \ldots, q'_l)$. Thus, by Proposition 9(a) we have $(INIT, COINIT, q_2, \ldots, q_k)$ is a pseudo-subsequence of $(q'_1, q'_2, \ldots, q'_l)$. But since $(HALF, q_2, \ldots, q_k)$ is a pseudo-subsequence of $(INIT, COINIT, q_2, \ldots, q_k)$ we have, $Q = (HALF, q_2, \ldots, q_k)$ is a pseudo-subsequence of $Q' = (q'_1, q'_2, q'_3, q'_4, \ldots, q'_l)$. A contradiction to the hypothesis. Part (b) follows.

(d) Let $Q''' = (q'_2, q'_3, \ldots, q'_l)$. We consider two cases.

Case 1: $q'_2 = HALF$.

By Proposition 9(a), Q is a pseudo-subsequence of Q' iff $(q_2, \ldots, q_k)$ is a pseudo-subsequence of Q'''. By Proposition 9(e), $(q_2, q_3, \ldots, q_k)$ is a pseudo-subsequence of Q''', iff $(COINIT, q_3, \ldots, q_k)$ is a pseudo-subsequence of Q'''. Thus, since Q is not a pseudo-subsequence of Q' it follows that $(COINIT, q_3, \ldots, q_k)$ is not a pseudo-subsequence of Q'''. Thus, by Proposition 9(a), it follows that $(q_1, COINIT, q_3, \ldots, q_k)$ is not a pseudo-subsequence of Q'.

Case 2: $q'_2 = INIT$.

By hypothesis and Proposition 9(a) we have that $(q_2, q_3, \ldots, q_k)$ is not a pseudo-subsequence of $(q'_2, q'_3, \ldots, q'_l)$. Thus, by part (b) $(COINIT, q_3, \ldots, q_k)$ is not a pseudo-subsequence of $(q'_2, q'_3, \ldots, q'_l)$. Thus, by Proposition 9(a) $(q_1, COINIT, q_3, \ldots, q_k)$ is not a pseudo-subsequence of Q'. $\square$

The following theorem shows that reducing any Q'-vector to its proper pseudo-subsequence Q results in the degree of learnability that is properly below the degree defined by the Q'-class.

Theorem 2. *Suppose* $Q = (q_1, \ldots, q_k) \in BASIC^k$ *and* $Q' = (q'_1, \ldots, q'_l) \in BASIC^l$. *Let* $R = R_1 \times R_2 \times \cdots \times R_k$, $R' = R'_1 \times R'_2 \times \cdots \times R'_l$, *where each* R_i (R'_i) *is an infinite subset of* N, *if* $q_i \in \{INIT, COINIT\}$ $(q'_i \in \{INIT, COINIT\})$, *and* R_i (R'_i) *is a subset of* Z, *with infinite intersection with both* N *and* Z^-, *if* $q_i = HALF$ $(q'_i = HALF)$.

If Q *is not a pseudo-subsequence of* Q' *then* $\mathscr{L}^{Q,R} \not\leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q',R'}$.

Proof of the above Theorem is given in Appendix A.

Corollary 1. *Suppose* $Q \in BASIC^k$ *and* $Q' \in BASIC^l$. *Then,* $\mathscr{L}^{Q} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$ *iff* Q *is a pseudo-subsequence of* Q'.

6. Definitions for open semi-hull and some propositions

Let **rat** denote the set of non-negative rationals. $\mathbf{rat}^+ = \mathbf{rat} - \{0\}$, denotes the set of positive rationals.

Any language $SEMI\_HULL^n_{\ldots}$ defined below is a geometrical figure semi-hull, collection of points in the first quadrant of the plane bounded by the y-axis and a broken line that consists of a straight fragment l_0 of the x-axis (starting from origin) followed by a straight fragment l_1 that makes an angle $\delta_1 < 90^\circ$ with the x-axis, followed by a fragment l_2 that makes an angle $\delta_2 > \delta_1$ with the x-axis, etc. (In the above, the angle is being measured anti-clockwise from the positive x-axis).

Definition 13. Suppose $a_1, \ldots, a_n \in N$ and $b_1, \ldots, b_n \in \mathbf{rat}^+$, where $0 < a_1 < a_2 < \ldots < a_n$.

$SEMI\_HULL^n_{a_1,b_1,a_2,b_2,\ldots,a_n,b_n} = \{(x,y) \in N^2 \mid y \geqslant \sum_{1 \leqslant i \leqslant n} b_i * (x \dot{-} a_i)\}$.

Note that $SEMI\_HULL^0 = N^2$. Also, note that though $SEMI\_HULL^n$ above are subsets of N^2, one can easily consider them as languages $\subseteq N$, by using pairing function. We assume such implicit coding whenever we are dealing with sets $\subseteq N^2$.

Parameters a_i in the above definition specify x-coordinates of break points of the border line, while the b_i specify the slopes that are being added to the slope of the border line after every break point.

To make our classes of languages learnable, we have to impose certain restrictions on the parameters a_i, b_i. First, we want both coordinates a and c of break points (a, c) to be integers. Secondly, for all languages in our classes, we fix a subset S from which slopes b_i may come from. (In the following definition, S may be an arbitrary subset of $\mathbf{rat}^+$; however, later we will impose additional restrictions on S). The definition of *valid* sequences of parameters a_i, b_i accomplishes this goal.

Definition 14. Suppose $a_1, \ldots, a_n \in N$ and $b_1, \ldots, b_n \in \mathbf{rat}^+$, where $0 < a_1 < a_2 < \cdots < a_n$. Suppose $S \subseteq \mathbf{rat}^+$.

We say that $(a_1, b_1, \ldots, a_n, b_n)$ is *valid* iff for $1 \leqslant j \leqslant n$, $[\sum_{1 \leqslant i \leqslant n} b_i * (a_j \dot{-} a_i)] \in N$. Additionally, if each $b_i \in S$, then we say that $(a_1, b_1, \ldots, a_n, b_n)$ is *S-valid*.

Let $VALID = \{(a_1, b_1, \ldots, a_n, b_n) \mid (a_1, b_1, \ldots, a_n, b_n)$ is valid$\}$.

Let $VALID_S = \{(a_1, b_1, \ldots, a_n, b_n) \mid (a_1, b_1, \ldots, a_n, b_n)$ is S-valid$\}$.

Note that the empty sequence () is both valid and S-valid. Also we require $a_1 > 0$. This is for technical convenience, and crucial for some of our results.

Now we define the class of languages we are going to explore.

Definition 15. Suppose $S \subseteq \mathbf{rat}^+$. Then
$SEMI\_HULL^{n,S} = \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \mid (a_1, b_1, \ldots, a_n, b_n) \in VALID_S\}$.

Now we formulate and prove a number of useful technical propositions (few of them immediately follow from the relevant definitions).

Proposition 11. (a) *Suppose* $(a_1, b_1, \ldots, a_j, b_j, a_{j+1}, b_{j+1}) \in VALID$. *Then*
$SEMI\_HULL^{j+1}_{a_1,b_1,\ldots,a_j,b_j,a_{j+1},b_{j+1}} \subset SEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j}$.

(b) *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \in VALID$, *and* $a_j \leqslant a'_j$, $b_j \geqslant b'_j$. *Then*, $SEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j} \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a'_j,b'_j}$.

Proof. Follows from definitions. □

Definition 16. Suppose $a_1, b_1, \ldots, , a_j, b_j$ are given such that $(a_1, b_1, \ldots, a_j, b_j) \in VALID$. Then, let
$INTER(a_1, b_1, \ldots, a_j, b_j) =$
$\bigcap\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n} \mid n \geqslant j \wedge (a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \in VALID\}$.

Intuitively, $INTER(a_1, b_1, \ldots, a_j, b_j)$, denotes the common portion of all $SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_j,b'_j,\ldots}$, with $a_i = a'_i$ and $b_i = b'_i$ for $1 \leqslant i \leqslant j$.

Proposition 12. (a) *Suppose* $1 \leqslant j \leqslant n$, *and* $(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \in VALID$. *Then* $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n}$.

(b) *Suppose* $(a_1, b_1, \ldots, a_j, b_j, a_{j+1}, b_{j+1})$ *is valid. Then,* $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq INTER(a_1, b_1, \ldots, a_j, b_j, a_{j+1}, b_{j+1})$.

Proof. Follows from definitions. □

Definition 17. Suppose $(a_1, b_1, \ldots, a_j, b_j) \in VALID$. Then let $maxinter(a_1, b_1, \ldots, a_j, b_j)$ denote the least natural number $x > a_j$ such that $\sum_{1 \leqslant i \leqslant j} b_i * (x \dot{-} a_i) \in N$.

Proposition 13. *Suppose* $a_1, b_1, \ldots, a_j, b_j$ *are given such that* $(a_1, b_1, \ldots, a_j, b_j) \in VALID$. *Then,* $INTER(a_1, b_1, \ldots, a_j, b_j) = SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j} \cap \{(x, y) \in N^2 \mid x \leqslant maxinter(a_1, b_1, \ldots, a_j, b_j)\}$.

Proof. Follows from definitions. □

Proposition 14. *Suppose* $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$, a_j, b_j, a'_j, b'_j *are given, where* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$, *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ *are valid. If* $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$.

Then

(i) $a'_j \leqslant a_j$ *and*

(ii) $b'_j * (maxinter(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \dot{-} a'_j) \geqslant b_j * (maxinter(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \dot{-} a_j)$.

Proof. Suppose that $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$, a'_j, b'_j, a_j, b_j are as given in the hypothesis.

If $a_j < a'_j$, then $(a'_j, \sum_{1 \leqslant i < j} b_i * (a'_j \dot{-} a_i))$ belongs to $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ but does not belong to $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$ (since $(a'_j, \sum_{1 \leqslant i < j} b_i * (a'_j \dot{-} a_i))$ is below the last fragment of the border line for $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$).

Thus, we must have $a'_j \leqslant a_j$.

Let $x = maxinter(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$. Let $y = b'_j * (x \dot{-} a'_j) + \sum_{1 \leqslant i < j} b_i * (x \dot{-} a_i))$. Now, (x, y) belongs to $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$. If (x, y) belongs to $SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$ then by definition of $SEMI\_HULL^j$, we must have $y = b'_j * (x \dot{-} a'_j) + \sum_{1 \leqslant i < j} b_i * (x \dot{-} a_i) \geqslant b_j * (x \dot{-} a_j) + \sum_{1 \leqslant i < j} b_i * (x \dot{-} a_i)$. Thus, $b'_j(x \dot{-} a'_j) \geqslant b_j(x \dot{-} a_j)$. The proposition follows. □

The following two corollaries have obvious geometrical interpretation.

Corollary 2. *Suppose* $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j, \dots, a'_n, b'_n)$ *and* $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j, \dots, a''_n, b''_n)$ *are valid.*

(a) *If* $a'_j < a''_j$, *then* $SEMI\_HULL^n_{a_1,b_1,\dots,a_{j-1},b_{j-1},a''_j,b''_j,\dots,a''_n,b''_n} \nsubseteq SEMI\_HULL^n_{a_1,b_1,\dots,a_{j-1},b_{j-1},a'_j,b'_j,\dots,a'_n,b'_n}$.

(b) *If* $a'_j = a''_j$ *and* $b'_j > b''_j$ *then* $SEMI\_HULL^n_{a_1,b_1,\dots,a_{j-1},b_{j-1},a''_j,b''_j,\dots,a''_n,b''_n} \nsubseteq SEMI\_HULL^n_{a_1,b_1,\dots,a_{j-1},b_{j-1},a'_j,b'_j,\dots,a'_n,b'_n}$.

Proof. If $SEMI\_HULL^n_{a_1,b_1,\dots,a_{j-1},b_{j-1},a''_j,b''_j,\dots,a''_n,b''_n} \subseteq SEMI\_HULL^n_{a_1,b_1,\dots,a_{j-1},b_{j-1},a'_j,b'_j,\dots,a'_n,b'_n}$, then it follows that $INTER(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a''_j, b''_j) \subseteq SEMI\_HULL^n_{a_1,b_1,\dots,a_{j-1},b_{j-1},a'_j,b'_j}$. Thus, it follows from Proposition 14 that $a''_j \leqslant a'_j$ and if $a''_j = a'_j$, then $b''_j \geqslant b'_j$. The corollary follows. □

Corollary 3. *Suppose* $(a_1, b_1, \dots, a_n, b_n)$ *and* $(a'_1, b'_1, \dots, a'_n, b'_n)$ *are valid. Suppose* $SEMI\_HULL^n_{a_1,b_1,\dots,a_n,b_n} \subset SEMI\_HULL^n_{a'_1,b'_1,\dots,a'_n,b'_n}$. *Let i be the minimum value such that* $a_i \neq a'_i$ *or* $b_i \neq b'_i$. *Then,* $a_i < a'_i$ *or* $[a_i = a'_i$ *and* $b_i > b'_i]$.

Proof. Note that for the least j such that $(a'_j, b'_j) \neq (a_j, b_j)$, we must have $INTER^n_{a_1,b_1,\dots,a_j,b_j} \subseteq SEMI\_HULL^n_{a_1,b_1,\dots,a_{j-1},b_{j-1},a'_j,b'_j}$. The corollary now follows from Proposition 14. □

Proposition 15. *Suppose* $1 \leqslant j \leqslant n$. *Suppose* $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a_j, b_j, \dots, a_n, b_n)$ *and* $(a_1, b_1, \dots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ *are valid.*

(a) *If* $a'_j \leqslant a_j$ *and* $b'_j \geqslant \sum_{j \leqslant i \leqslant n} b_i$, *then* $SEMI\_HULL^j_{a_1,b_1,\dots,a_{j-1},b_{j-1},a'_j,b'_j} \subseteq SEMI\_HULL^n_{a_1,b_1,\dots,a_j,b_j,\dots,a_n,b_n}$.

(b) *If* $a'_j \leqslant a_j$ *and* $b'_j > \sum_{j \leqslant i \leqslant n} b_i$, *then* $SEMI\_HULL^j_{a_1,b_1,\dots,a_{j-1},b_{j-1},a'_j,b'_j} \subset SEMI\_HULL^n_{a_1,b_1,\dots,a_j,b_j,\dots,a_n,b_n}$.

Proof. (a) $SEMI\_HULL^j_{a_1,b_1,\dots,a_{j-1},b_{j-1},a'_j,b'_j} = \{(x,y) \in N^2 \mid y \geqslant b'_j(x \dot{-} a'_j) + \sum_{1 \leqslant i < j} b_i * (x \dot{-} a_i)\}$.

$SEMI\_HULL^n_{a_1,b_1,\dots,a_n,b_n} = \{(x,y) \in N^2 \mid y \geqslant \sum_{j \leqslant i \leqslant n} b_i * (x \dot{-} a_i) + \sum_{1 \leqslant i < j} b_i * (x \dot{-} a_i)\}$.

Part (a) follows since $b'_j(x \dot{-} a'_j) \geqslant b'_j(x \dot{-} a_j) \geqslant \sum_{j \leqslant i \leqslant n} b_i * (x \dot{-} a_j) \geqslant \sum_{j \leqslant i \leqslant n} b_i * (x \dot{-} a_i)$.

(b) Follows from part (a) and Corollary 2. □

Now we are at the point when our results require additional constraint on the set S (of slopes). Intuitively, the set S satisfying the constraints cover the positive rational numbers, and can be algorithmically listed in a monotonic order on a two-sided-infinite tape. A natural example of set S satisfying the constraint below is the set $N \cup \{1/x | x \in N\}$. Although our results below hold for any fixed set S of rationals satisfying the constraint in question, we suggest the reader to keep in mind the above set when reading the proofs.

Definition 18. A set $S \subseteq \mathbf{rat}^+$ is said to be $\mathbf{rat}^+$-*covering* iff there exists a recursive bijection f from Z to S such that,

(i) for $i, j \in Z$, $i < j$ iff $f(i) < f(j)$.
(ii) for every $x \in \mathbf{rat}^+$, there exist $y, y' \in S$ such that $y < x < y'$.

A natural choice for a set S (which does not satisfy the above constraint) seems to be the set $\mathbf{rat}^+$. However, in this case, a complete class of languages $\{L_y = \{x | x \in \mathbf{rat}^+, x \geqslant y\} \mid y \in \mathbf{rat}^+\}$ (see [JKW00]) would be trivially reducible to any class of languages-figures considered in our paper, thus making all of them of the same complexity. The use of $\mathbf{rat}^+$-*covering* sets S gives us opportunity to capture differences in learnability of different geometrical concepts observed in our paper.

Our results below hold for any $\mathbf{rat}^+$-*covering* set S. However, it is open at present whether $SEMI\_HULL^{n,S} \equiv^{\mathbf{TxtEx}} SEMI\_HULL^{n,S'}$, for arbitrary $\mathbf{rat}^+$-*covering* sets S and S'.

Note that, for any given $n \in N$, and $\mathbf{rat}^+$-*covering* set S, there exists a limiting partial recursive function F such that, if i is a grammar for some $SEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n}$, where $(a_1, b_1, \ldots, a_n, b_n)$ being S-valid, then $F(i)$ converges to $(a_1, b_1, \ldots, a_n, b_n)$. That is one can determine the parameters limit-effectively from any grammar for $SEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n}$. This fact will be implicitly used in construction of Ψ in some of our proofs.

Intuitively, the following technical proposition claims that, given a point (x_0, y_0) in a language $SEMI\_HULL^{j-1}_{a_1,b_1,\ldots,a_{j-1},b_{j-1}}$, one can effectively find a lower bound on the boundary slopes b_j of all languages $SEMI\_HULL^{j}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,\ldots}$ which do not contain (x_0, y_0).

Proposition 16. *Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ is valid. Let $(x_0, y_0) \in N^2$ be such that $y_0 > \sum_{1 \leqslant i < j} b_i(x_0 \dot{-} a_i)$. Then, there exists a $B' \in \mathbf{rat}^+$ obtainable effectively from $a_1, b_1, \ldots, a_{j-1}, b_{j-1}, x_0$ and y_0 such that for any (a'_j, b'_j), if $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, is valid and $x_0 \leqslant a'_j$ or $b'_j < B'$, then, $(x_0, y_0) \in INTER^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$.*

Proof. If $a'_j \geqslant x_0$, then we would have $y_0 \geqslant \sum_{1 \leqslant i < j} b_i(x_0 \dot{-} a_i) = b'_j(x_0 \dot{-} a'_j) + \sum_{1 \leqslant i < j} b_i(x_0 \dot{-} a_i)$. Thus, $(x_0, y_0) \in INTER^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$.

Now fix any $a'_j < x_0$. If $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is valid, but $(x_0, y_0) \notin INTER^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$, then either (a) $maxinter(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < x_0$, or (b) $y_0 < b'_j(x_0 \dot{-} a'_j) + \sum_{1 \leqslant i < j} b_i(x_0 \dot{-} a_i)$.

Let $B(a'_j) = (y_0 - \sum_{1 \leqslant i < j} b_i(x_0 \dot{-} a_i))/(x_0 \dot{-} a'_j)$. Now for (b) to be true b'_j must be greater than $B(a'_j)$. We now consider which $b'_j \leqslant B(a'_j)$ can satisfy (a). Let $x_1 = maxinter(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, and $y_1 = b'_j * (x_1 \dot{-} a'_j) + \sum_{1 \leqslant i < j} b_i * (x_1 \dot{-} a_i)$. For $b'_j \leqslant B(a'_j)$ to satisfy (a), (x_1, y_1) must lie in the intersection of the following three regions:

(A) $y > \sum_{1 \leqslant i < j} b_i(x \dot{-} a_i)$,
(B) $x \leqslant x_0$, and
(C) $y \leqslant B(a'_j) * (x \dot{-} a'_j) + \sum_{1 \leqslant i < j} b_i(x \dot{-} a_i)$.

Since the above intersection is finite, there are only finitely many possibilities for (x_1, y_1), and thus for b'_j. Thus, let $B_1(a'_j) \leqslant B(a'_j)$, be a positive rational number, such that $maxinter(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < x_0$ implies $b'_j > B_1(a'_j)$. Note that such a $B_1(a'_j)$ exists and can be obtained effectively from $a_1, b_1, \ldots, a_{j-1}, b_{j-1}, x_0, y_0$.

Now taking $B' = \min(\{B_1(a'_j) \mid a_{j-1} < a'_j < x_0\})$, witnesses the proposition. $\square$

Corollary 4. *Suppose $(a_1, b_1, \ldots, a_j, b_j)$ is valid. Then, one can effectively (in $a_1, b_1, \ldots, a_j, b_j$) obtain a $B' \in \mathbf{rat}^+$ such that following property is satisfied.*

For any $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ which is valid, if $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^n_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$, then $b'_j \geqslant B'$ and $a'_j \leqslant a_j$.

Proof. Pick $(x_0, y_0) \in N^2$ such that $x_0 > a_j$, and $\sum_{1 \leqslant i < j} b_i(x_0 \dot{-} a_i) < y_0 < \sum_{1 \leqslant i \leqslant j} b_i(x_0 \dot{-} a_i)$. Corollary, now follows from Propositions 14 and 16. $\square$

Proposition 18 below will play an important role in some of our proofs. Imagine that, in the process of learning, $SEMI\_HULL^n_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j, \ldots, a_n, b_n}$, the learner has already learned the parameters $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$, and is now trying to learn the parameters a_j, b_j. Our aim is to use a $(INIT, COINIT)$-type strategy to learn (a_j, b_j) (instead of the trivial $(INIT, HALF)$-type strategy mentioned in the introduction).

Since we intend to use $(INIT, COINIT)$-type strategy, we need to be safe in choosing the parameters. That is, choosing (a_j, b_j) must imply that any (a'_j, b'_j) smaller than (a_j, b_j) must be inconsistent with the input, (smaller in the sense of some $(INIT, COINIT)$-type ordering of all pairs (a''_j, b''_j)).

This is achieved in two steps. First we order (a''_j, b''_j), with $b''_j \leqslant B$, for some fixed positive rational number B in a $INIT$ like fashion. Then, we order $b''_j > B$, in a $COINIT$ like fashion. This latter ordering is easily achievable (as will be seen in the proof of Theorem 3 below). For the former ordering, if $b'_j, b_j \leqslant B$, then we make sure that if $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is a subset of $SEMI\_HULL^n_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$, then ordering places (a'_j, b'_j) below (a_j, b_j). Proposition 18 (which is based on Corollary 5 and Proposition 17 below) shows that this is achievable. Note that this ordering depends on $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$.

Corollary 5. *Suppose $1 \leqslant j \leqslant n$. Let S be any $\mathbf{rat}^+$-covering set, and $B \in S$.*

Suppose $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$ a_j, b_j are given, where $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ is S-valid.

Then, there exist only finitely many (a'_j, b'_j), such that

(i) *$(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is S-valid,*

(ii) *$b'_j \leqslant B$, and*

(iii) *$INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$.*

Moreover, canonical index for the finite set of a'_j, b'_j *satisfying above three conditions can be obtained effectively from* $B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j$.

Furthermore, for any a'_j, b'_j *satisfying the above three conditions,* $a'_j \leqslant a_j$, *and if* $a'_j = a_j$, *then* $b'_j \geqslant b_j$.

Proof. Furthermore clause follows from Proposition 14.

By Corollary 4, one can effectively find a $B' \in \mathbf{rat}^+$ such that any (a'_j, b'_j) satisfying (i) and (iii) must satisfy $b'_j \geqslant B'$ and $a'_j \leqslant a_j$. Corollary now follows since there are only finitely many (a'_j, b'_j) such that $a'_j \leqslant a_j$, $B' \leqslant b'_j \leqslant B$, and $b'_j \in S$ and for any (a'_j, b'_j), one can effectively test whether (a'_j, b'_j) satisfies clauses (i) to (iii) in the corollary. □

The above corollary together with the following technical proposition will enable us to impose a suitable ordering on $(a_1, b_1, \ldots, a_j, b_j)$ with an upper bound B on slopes b_j. Think of $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ as already fixed.

Proposition 17. *Suppose R is a partial order over an r.e. set A, and F is a partial recursive function with domain A such that, for any* $x \in A$,

(i) $\{x' \mid x'Rx\}$ *is finite and*
(ii) $F(x)$ *is the canonical index for* $\{x' \in A \mid x'Rx\}$.

Then, effectively from a program for F, one can obtain a program for a 1–1, *partial recursive function h with domain A and range* $\subseteq N$, *such that* $x'Rx$ *implies* $h(x') \leqslant h(x)$. *Moreover, if A is infinite, then range of h is N.*

Proof. Let f be a recursive function such that $range(f) = A$. Define h as follows. Let $cur = 0$. Go to stage 0.

Stage s
(* In this stage we make sure that $h(f(s))$, along with $h(x)$, for all x such that $xRf(s)$, are defined appropriately. *)

If $h(f(s))$ has already been defined, then go to stage $s+1$.
Otherwise let $X = \{x \mid xRf(s)\}$ (note that one can effectively obtain X from $f(s)$ by assumption (ii) of the hypothesis; also note that $f(s) \in X$, since $f(s)Rf(s)$).
Let $x_0, x_1, \ldots, x_m$ be listing of members of X such that $x_i R x_j$ implies $i \leqslant j$ (note that one can easily get such an ordering since X is finite).
For $j = 0$ to m do
 If $h(x_j)$ has not been defined then
 let $h(x_j) = cur$; $cur = cur + 1$.
EndFor
(* Note that $h(f(s))$ along with $h(x)$, for all x such that $xRf(s)$, have been defined. *)

End Stage s

It is easy to verify that h satisfies the properties required in the proposition. $\square$

Proposition 18. *Let S be any* $\mathbf{rat}^+$*-covering set. Then, there exists a recursive function code with domain* $\mathbf{rat} \times VALID_S$, *and range* $\subseteq N$ *such that following is satisfied.*

Suppose $1 \leqslant j \leqslant n$, $B \in S$, *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ *is S-valid.*

(A) *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ *are S-valid.*

(A.1) *If* $b_j, b'_j \leqslant B$ *and* $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$, *then* $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \leqslant code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$.

(A.2) *If* $b_j, b'_j \leqslant B$ *and* $(a_j, b_j) \neq (a'_j, b'_j)$, *then* $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \neq code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$.

(A.3) $\{code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j) \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j) \in VALID_S$ *and* $b''_j \leqslant B\} = N$.

(B) *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$ *is* S-valid. *If* $b_j > B$, *then* $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B)$.

Proof. Fix $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$ and B as in the hypothesis. Let us first define a relation Rel on $N \times S \cap \{r \in \mathbf{rat} \mid r \leqslant B\}$ as follows:

$$(a', b') Rel (a, b) \text{ iff} INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a', b') \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a,b}.$$

Note that (by Corollary 5) for each (a, b), there are only finitely many (a', b') such that $(a', b') Rel (a, b)$. Also, for each of these (a', b'), $a' \leqslant a$ and either $a' < a$ or $b' \geqslant b$. Furthermore, canonical index for $\{(a', b') \mid (a', b') Rel (a, b)\}$, can be effectively obtained from (a, b).

Let Rel^* be transitive closure of Rel. It is easy to verify that Rel^* is a partial order, where for each (a, b), there exists at most finitely many (a', b') such that $(a', b') Rel^* (a, b)$. Moreover, canonical index for $\{(a', b') \mid (a', b') Rel^* (a, b)\}$, can be effectively obtained from (a, b).

Existence of *code* as required now follows from Proposition 17. $\square$

7. Q-classes to which $SEMI\_HULL^{n,S}$ is reducible

Our goal in this section is to establish an upper bound on the $SEMI\_HULL^{n,S}$ degrees in terms of the Q-hierarchy. To find such a bound, we actually have to design a learning strategy for languages in $SEMI\_HULL^{n,S}$ that consists of q_i-type strategies for some $Q = (q_1, q_1, \ldots, q_k)$, and a grammar learned by every q_i is used by q_{i+1}. A natural strategy of this type would be the following $(q_1, q_2, \ldots, q_{2n-1}, q_{2n})$-strategy, where $q_{2i+1} = INIT$ and $q_{2i+2} = HALF$ for $i < n$: learn the first break point a_1 using an $INIT$-type strategy; once a_1 has been learned, learn the first slope b_1 at the point $(a_1, 0)$ using a $HALF$-type strategy; then learn the second break point $(a_2, b_1 * (a_2 - a_1))$ using an $INIT$-type strategy, etc. However, a much more efficient learning strategy is suggested by the following. Informally one can visualize this strategy as follows. Assume that slope values come from the set $N \cup \{1/n \mid n \in N\}$. It may happen that in the beginning the learner receives points (x, y) indicating that the slope to be learned is greater or equal 1. Then the learner uses an *INIT*-like strategy to learn a break and a *COINIT*-like strategy (not a *HALF*-strategy as above!) to learn the slope: the slopes tend to 1 from above, and the learner uses this assumption. If the slope gets smaller than 1, the learner then uses a combined *INIT*-like strategy to learn the break point and the slope together: both of them change now in *INIT*-fashion.

There is a slight problem though in the above strategy. It may be possible that slope at some point seems less than 1, but later on when lots of new points $(i, 0)$ come from the input, slope again seems larger than 1. To prevent this from harming the learning process, the learner uses the combined *INIT*-strategy in a *safe* fashion: Informally, suppose one has learned the parameters, $a_1, b_1, \ldots, a_{j-1}, b_{j-1}$, and is now trying to determine a_j, b_j. Now we need to make sure that the combined *INIT*-strategy does not commit to (a_j, b_j) being (a'_j, b'_j) before it has been able to determine that input data cannot be from $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j,\ldots}$, for any other $a''_j, b''_j, \ldots$, which satisfies $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a''_j,b''_j,\ldots} \subset SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j,a'_{j+1},b'_{j+1},\ldots}$, for some value of the parameters $a'_{j+1}, b'_{j+1}, \ldots$. It is possible to achieve this by using Proposition 18.

The actual proof of the theorem technically looks somewhat different, and the above method is a bit hidden.

Theorem 3. *Suppose S is* **rat**$^+$*-covering. Suppose* $Q = (q_1, q_2, \ldots, q_{2n-1}, q_{2n})$, *where* $q_{2i+1} = INIT$ *and* $q_{2i+2} = COINIT$, *for* $i < n$. *Then* $SEMI\_HULL^{n,S} \leqslant^{\mathbf{TxtEx}} \mathcal{L}^Q$.

Proof. Fix S which is **rat**$^+$*-covering*. Let h be a recursive bijection from Z to S such that $h(i) < h(i+1)$. Let $B = h(0)$.

The intuitive idea of the learning strategy is as follows. Suppose we have already learned $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$. Then, we use *INIT* like strategy to learn any pair (a_j, b_j), if $b_j \leqslant B = h(0)$, and use *COINIT*-type strategy to learn b_j if $b_j > B$. The former is done using *code* as in Proposition 18. The latter can be done easily by using h to form a *COINIT*-like strategy.

Going through the proof below (and the proofs for other upper bounds in this paper) one must be aware of the fact that, though intuitively we do use *INIT*/*COINIT* (or whatever—in other proofs) strategy for learning each successive parameter, in the actual proof we do it by choosing an appropriate minimal consistent *SEMI_HULL* on every step and *INIT*/*COINIT* strategy is in some sense hidden.

Now we proceed with the formal construction.

Fix *code* as in Proposition 18. Let g be a function from S to N such that $g(b) = 0$, if $b \leqslant B$, and $g(b) = h^{-1}(b)$, otherwise. (Note that g is monotonically non-decreasing).

The following claim utilizes the construction of Proposition 18 to show that the values $(code(B, a_1, b_1, \ldots, a_j, b_j))$ can be used as conjectures by *INIT*-type substrategies, and the values $g(b_j)$ can be used as conjectures by *COINIT*-type substrategies.

Claim 1. *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j), (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \in VALID_S$, *where* $(a_j, b_j) \neq (a'_j, b'_j)$. *Suppose* $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) \subset SEMI\_HULL^n_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j}$.

Then,

(i) $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) < code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, *or*

(ii) $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, *and* $g(b_j) > g(b'_j)$.

Proof. By Proposition 14 we have that $a_j \leqslant a'_j$.

We now consider following cases:

Case 1: $b_j, b'_j \leqslant B$.

Since, $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j}$, we have $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) < code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18; for getting $<$ instead of $\leqslant$ use the fact that $(a_j, b_j) \neq (a'_j, b'_j)$).

Case 2: $b_j \leqslant B < b'_j$.

In this case $a_j < a'_j$ by Proposition 14.

Also, $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j} \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B}$.

Thus, $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) < code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18; for getting $<$ instead of $\leqslant$ use the fact that $(a_j, b_j) \neq (a'_j, B)$).

Case 3: $b'_j \leqslant B < b_j$.

In this case, $INTER(a_1, b_1, \ldots, a_j, B) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B} \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j}$. Thus, $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B) \leqslant code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18). Also $g(b'_j) = 0 < g(b_j)$.

Case 4: $B < b_j \leqslant b'_j$.

In this case $a_j < a'_j$ by Proposition 14. Thus, $INTER(a_1, b_1, \ldots, a_j, B) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B} \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B}$. Thus, $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B) < code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18; for getting $<$ instead of $\leqslant$ use the fact that $(a_j, B) \neq (a'_j, B)$).

Case 5: $B < b'_j \leqslant b_j$.

If $a_j < a'_j$, then $INTER(a_1, b_1, \ldots, a_j, B) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B} \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B}$. Thus, $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B) < code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18; for getting $<$ instead of $\leqslant$ use the fact that $(a_j, B) \neq (a'_j, B)$).

If $a_j = a'_j$, then $b'_j < b_j$ (by Proposition 14). Thus, $g(b_j) > g(b'_j)$. Also, $INTER(a_1, b_1, \ldots, a_j, B) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B}$. Thus, $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, B) \leqslant code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, B) = code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ (by Proposition 18). $\square$

Let *map* be a mapping from $VALID$ to N^* such that
$map(a_1, b_1, \ldots, a_n, b_n) = (code(B, a_1, b_1), g(b_1),$
$code(B, a_1, b_1, a_2, b_2), g(b_2), \ldots, code(B, a_1, b_1, \ldots, a_n, b_n), g(b_n))$.

Now we suggest the reader to recall the properties of the ordering $<_Q$. For example, note that if $Q = (INIT, COINIT)$ then $(i,j) <_Q (i',j')$ would mean that $i < i'$ or ($i = i'$ and $j > j'$).

Claim 2. *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$ *and* $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *are S-valid.*

(A) *If* $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, *then, for the least* j *such that* $(a_j, b_j) \neq (a'_j, b'_j)$, $INTER(a_1, b_1, \ldots, a_j, b_j) \nsubseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$.

(B) *If* $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, *Then*, $INTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \nsubseteq SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

(C) *If* $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subset SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$, *then* $map(a_1, b_1, \ldots, a_n, b_n) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n)$.

Proof. (A) Let j be least number such that $(a_j, b_j) \neq (a'_j, b'_j)$. Note that, for $i < j$, we must have $code(B, a_1, b_1, \ldots, a_i, b_i) = code(B, a'_1, b'_1, \ldots, a'_i, b'_i)$, and $g(b_i) = g(b'_i)$. If $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$, then, by Claim 1 we would have $code(B, a_1, b_1, \ldots, a_j, b_j) < code(B, a'_1, b'_1, \ldots, a'_j, b'_j)$ or $code(B, a_1, b_1, \ldots, a_j, b_j) = code(B, a'_1, b'_1, \ldots, a'_j, b'_j)$ and $g(b_j) > g(b'_j)$. Thus, $map(a_1, b_1, \ldots, a_n, b_n) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n)$, a contradiction to the hypothesis.

(B) Let j be the least number such that $(a_j, b_j) \neq (a'_j, b'_j)$. Now (B) follows using part (A) and the fact that $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq INTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n)$ and $SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_j,b'_j,\ldots,a'_n,b'_n} \subseteq SEMI\_HULL^j_{a'_1,b'_1,\ldots,a'_j,b'_j}$.

(C) Follows from part (B) and $INTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$. $\square$

We now continue with the proof of the theorem. The aim is to construct Θ which maps $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ to $L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$ to a grammar sequence converging to a grammar for $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ would be trivial. We thus just define Θ.

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $Prop(X, a_1, b_1, \ldots, a_n, b_n)$ be true iff following two properties are satisfied.

(A) $(a_1, b_1, \ldots, a_n, b_n) \in VALID_S$.

(B) For all $(a'_1, b'_1, \ldots, a'_n, b'_n) \in VALID_S$ such that $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, $X \nsubseteq SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Note that condition (B) above is equivalent to
(B$'$) For all j, $1 \leqslant j \leqslant n$, (B$'$.1) and (B$'$.2) are satisfied.
(B$'$.1) For all $a'_j \in N, b'_j \in S$, $b'_j \leqslant B$,
if $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j))$, then
$[X \nsubseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}]$.
(B$'$.2) If $B \leqslant b_j$, then for all $b'_j \in S$ such that $b_j < b'_j$, $X \nsubseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j}$.
Note that B$'$.2 above is equivalent to
(B$''$.2) If $B \leqslant b_j$, then for least $b'_j \in S$ such that $b_j < b'_j$, $X \nsubseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j}$.

Note that whether $X, a_1, b_1, \ldots, a_n, b_n$, satisfy (A) and (B$'$.1) and (B$''$.2) for all j, $1 \leqslant j \leqslant n$, is effectively testable.

Thus, for any finite set $X \subseteq N^2$, let

$$\Theta(X) = \bigcup \{ L^Q_{map(a_1,b_1,\ldots,a_n,b_n)} \mid Prop(X, a_1, b_1, \ldots, a_n, b_n) \}.$$

For infinite X', $\Theta(X') = \bigcup_{X \subseteq X',\, \mathrm{card}(X) < \infty} \Theta(X)$.
It is easy to verify that
(1) for any $X \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$,

$$\Theta(X) \subseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$$

(due to clause (B) in definition of *Prop* above, and the fact that for any valid I and I', $map(I) <_Q map(I')$, implies $L^Q_{map(I)} \subseteq L^Q_{map(I')}$), and

(2) for any finite set $X \subseteq N^2$ such that $\{(x, y) \in N^2 \mid x \leqslant maxinter(a_1, b_1, \ldots, a_n, b_n)$ and $y = \min(\{y' | (x, y') \in INTER(a_1, b_1, \ldots, a_n, b_n)\})\} \subseteq X \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$,

$$\Theta(X) \supseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}.$$

(By Claim 2(B), and definition of *Prop* and Θ).
Thus, we have that $\Theta(SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}) = L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$. □

Now we will show that the above theorem is in some sense optimal. That is, for $Q = (q_1, q_2, \ldots, q_{2n-1}, q_{2n})$, where $q_{2i+1} = INIT$, and $q_{2i+2} = COINIT$, for $i < n$, and any $Q' \in BASIC^*$, if $\mathscr{L}^Q \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$, then $SEMI\_HULL^{n,S} \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$. Thus, Q in the above theorem cannot be improved if we use components only from *BASIC* (whether we can improve it by using some other basic components is open).

Theorem 4. *Suppose $n \in N^+$, $1 \leqslant j \leqslant n$. Suppose S is any* **rat**$^+$*-covering set.*
(1) *Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ is S-valid.*
Let $Q = (q_1, q_2, \ldots, q_{2(n-j+1)})$, where $q_{2i+1} = INIT$ and $q_{2i+2} = COINIT$ (for $i \leqslant n - j$). Suppose $Q' \in BASIC^*$ is such that $\mathscr{L}^Q \nleqslant \mathscr{L}^{Q'}$.
Then
$\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j\ldots,a'_n,b'_n} | (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j \ldots, a'_n, b'_n)$ *is S-valid*$\} \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$.

(2) *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j)$ *is such that there exists a* b_j, *such that* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ *is S-valid.*

Let $Q = (q_1, \ldots, q_{2(n-j)+1})$, *where* $q_{2i+1} = COINIT$, *for* $i \leqslant n-j$, *and* $q_{2i+2} = INIT$, *for* $i < n-j$. *Suppose* $Q' \in BASIC^*$ *is such that* $\mathscr{L}^Q \nleqslant \mathscr{L}^{Q'}$.

Then $\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n} | (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j \ldots, a'_n, b'_n)$ *is S-valid*$\}$ $\nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$.

Proof. The intuitive idea of the proof is to use a's to diagonalize against sequences of $COINIT$ and b's to diagonalize against sequences of $INIT$.

For a fixed n, we prove the above theorem by reverse induction on j (from $j = n$ to $j = 1$). For each such j, we first show (2) and then (1).

Base Case: $j = n$, *for* (2):

In this case $Q = (COINIT)$.

Now $COINIT \leqslant^{\mathbf{TxtEx}} \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,b'_n} | (a_1, b_1, \ldots, a_{n-1}, b_{n-1}, a_n, b'_n)$ is S-valid$\}$ as witnessed by Θ and Ψ defined as follows.

Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

For any set $Y \subseteq N$, let $\Theta(Y) = \bigcup_{i \in Y} SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,h(i)}$.

It is easy to verify that $\Theta(L^Q_i) = SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,h(i)}$. Let Ψ be defined as follows. If a sequence α of grammars converges to a grammar for $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,h(i)}$, then $\Psi(\alpha)$ converges to a grammar for L^Q_i.

It is easy to verify that Θ and Ψ witness that
$COINIT \leqslant^{\mathbf{TxtEx}} \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{n-1},b_{n-1},a_n,b'_n} | (a_1, b_1, \ldots, a_{n-1}, b_{n-1}, a_n, b'_n)$ is S-valid$\}$.

Thus, since $COINIT \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$, (2) is proven for $j = n$.

Induction Case (*for* (1)):

Suppose for $j > j'$, we have shown both (1) and (2), and for $j = j'$, we have shown (2). Then we show (1) for $j = j'$.

Note that $Q = (q_1, q_2, \ldots, q_{2(n-j+1)})$, where q_{2i+1} is $INIT$ and q_{2i+2} is $COINIT$, for $i \leqslant n-j$. Without loss of generality assume that Q' contains at least one $INIT$ or $HALF$ (otherwise just add $INIT$ at the end of Q'). Suppose $Q' = (q'_1, \ldots, q'_{k-1}, q'_k, q'_{k+1}, \ldots, q'_l)$, where $q'_1, \ldots, q'_{k-1}$ are $COINIT$, q'_k may be either $INIT$ or $HALF$, and for $k+1 \leqslant i \leqslant l$, $q'_i \in \{INIT, COINIT, HALF\}$. Without loss of generality assume that q'_k is $INIT$ (since otherwise we could just replace $q'_k = HALF$ with $(COINIT, INIT)$ and consider $Q' = (q'_1, \ldots, q'_{k-1}, COINIT, INIT, q'_{k+1}, \ldots, q'_l)$).

Now, by hypothesis we know that $\mathscr{L}^Q \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$. Thus, by Corollary 1, Q is not a pseudo-subsequence of Q'. Now consider Q'' obtained from Q by dropping $q_1 = INIT$, Q''' obtained from Q' by dropping $q'_1, q'_2, \ldots, q'_{k-1}$. Now, Q is not a pseudo-subsequence of Q''' by repeated use of Proposition 9(b), and thus Q'' is not a pseudo-subsequence of Q''' by Proposition 10(a). Thus, by Corollary 1, $\mathscr{L}^{Q''} \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'''}$.

Now, suppose by way of contradiction that Θ along with Ψ witnesses the reduction

$$\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j\ldots,a'_n,b'_n} | (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j \ldots, a'_n, b'_n) \text{ is } S\text{-valid } \}$$
$$\leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}.$$

Recall that $SEMI\_HULL^0$ is N^2. Let
$v_1 = \min(\{u_1 | (\exists u_2, \ldots, u_l)[\langle u_1, u_2, \ldots, u_l \rangle \in \Theta(SEMI\_HULL^{j-1}_{a_1,b_1,\ldots,a_{j-1},b_{j-1}})]\})$.

For $1 < i < k$, let
$v_i = \min(\{u_i | (\exists u_{i+1}, \ldots, u_l)[\langle v_1, \ldots, v_{i-1}, u_i, \ldots, u_l \rangle \in \Theta(SEMI\_HULL^{j-1}_{a_1,b_1,\ldots,a_{j-1},b_{j-1}})]\})$.

Let σ be such that $content(\sigma) \subseteq SEMI\_HULL^{j-1}_{a_1,b_1,\ldots,a_{j-1},b_{j-1}}$, and there exist $u_k, \ldots, u_l$, such that $\langle v_1, v_2, \ldots, v_{k-1}, u_k, \ldots, u_l \rangle \in \Theta(\sigma)$.

Let $a_j > \max(\{x \in N | (\exists y \in N)[\langle x, y \rangle \in content(\sigma)]\})$ be such that, for some b_j, $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ is S-valid. Note that, for all $b''_j, a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n$, such that $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b''_j, a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n)$ is S-valid, $content(\sigma) \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b''_j,a''_{j+1},b''_{j+1},\ldots,a''_n,b''_n}$. Thus, Θ (along with Ψ) essentially witnesses that

$$\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n} | (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j \ldots, a'_n, b'_n) \text{ is } S\text{-valid } \}$$
$$\leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'''}$$

(since parameters for the first $k-1$ *COINIT*s in Q' are fixed to be $v_1, \ldots, v_{k-1}$). This is not possible by induction hypothesis (since $\mathscr{L}^{Q''} \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'''}$).

Induction Case (*for* (2)): Suppose for $j > j'$, we have shown both (1) and (2). Then we show (2) for $j = j'$.

In this case, $Q = (q_1, q_2, \ldots, q_{2(n-j)+1})$, where q_{2i+1} is *COINIT*, for $i \leqslant n-j$, and q_{2i+2} is *INIT*, for $i < n-j$. Without loss of generality assume that Q' contains at least one *COINIT* or *HALF* (otherwise just add *COINIT* to the end of Q'). Suppose Q' is of form $(q'_1, \ldots, q'_{k-1}, q'_k, \ldots, q'_l)$, where $q'_1, \ldots, q'_{k-1}$ are *INIT*, q'_k may be either *COINIT* or *HALF*, and for $k+1 \leqslant i \leqslant l$, $q'_i \in \{INIT, COINIT, HALF\}$. Without loss of generality assume that q'_k is *COINIT* (since otherwise we could just replace $q'_k = HALF$ with $(INIT, COINIT)$ and consider $Q' = (q'_1, \ldots, q'_{k-1}, INIT, COINIT, q'_{k+1}, \ldots, q'_l)$).

Now, by hypothesis we know that $\mathscr{L}^Q \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$. Thus, by Corollary 1, Q is not a pseudo-subsequence of Q'. Thus, for Q'' obtained from Q by dropping $q_1 = COINIT$, Q''' obtained from Q' by dropping $q'_1, q'_2, \ldots, q'_{k-1}$ (note that q'_k is *COINIT*, while the first q in Q'' is *INIT*), Q'' is not a pseudo-subsequence of Q''' (this case is similar to the one in Induction Case for (1) with *INIT*s and *COINIT*s reversed). Thus, by Corollary 1, $\mathscr{L}^{Q''} \nleqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'''}$.

Suppose by way of contradiction that Θ along with Ψ witnesses the reduction

$$\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n} | (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j \ldots, a'_n, b'_n) \text{ is } S\text{-valid } \}$$
$$\leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}.$$

Let $X = \{\langle u_1, u_2, \ldots, u_l \rangle | (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b'_j \ldots, a'_n, b'_n)$ is S-valid, and $\Theta(SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n}) = L^{Q'}_{u_1,u_2,\ldots,u_l}\}$.

Let $v_1 = \min(\{u_1 | (\exists u_2, \ldots, u_l) \langle u_1, u_2, \ldots, u_l \rangle \in X\})$.

For $1 < i < k$, let $v_i = \min(\{u_i \mid (\exists u_{i+1}, \ldots, u_l) \langle v_1, \ldots, v_{i-1}, u_i, \ldots, u_l \rangle \in X\})$.

Let $b'_j, \ldots, a'_n, b'_n$ be such that $\Theta(SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n}) = L^{Q'}_{v_1,\ldots,v_{k-1},u_k,\ldots,u_l}$ for some $u_k, \ldots, u_l$.

Let us fix a $b_j > \sum_{k=j}^{k=n} b'_k$, $b_j \in S$. Then, for any a''_w, b''_w, $j < w \leqslant n$, such that $(a_1, b_1, \ldots, a_j, b_j, a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n)$ is S-valid, we have that

$$\Theta(SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,a''_{j+1},b''_{j+1},\ldots,a''_n,b''_n}) = L^{Q'}_{v_1,\ldots,v_{k-1},u_k,\ldots,u_l},$$

for some values of $u_k, \ldots, u_l$ (due to monotonicity of Θ; note that $SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j} \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n}$ by Proposition 15, and, therefore, $SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,a''_{j+1},b''_{j+1},\ldots,a''_n,b''_n} \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b'_j\ldots,a'_n,b'_n}$, but $v_1, \ldots, v_{k-1}$ are chosen to be the minimum possible values, thus only $u_k, \ldots, u_l$ can vary for $a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n$).

Thus, Θ (along with Ψ) essentially witnesses that

$$\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j,a''_{j+1},b''_{j+1},\ldots,a''_n,b''_n} | (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j, a''_{j+1}, b''_{j+1}, \ldots, a''_n, b''_n) \text{ is } S\text{-valid } \} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'''}$$

(since parameters for the first $k-1$ $INIT$s are fixed in Q' to be $v_1, \ldots, v_{k-1}$). This is not possible by induction hypothesis (since $\mathscr{L}^{Q''} \not\leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'''}$). $\square$

Corollary 6. *Suppose S is* $\mathbf{rat}^+$*-covering. Let $Q = (q_1, \ldots, q_{2n})$, where $q_{2i+1} = INIT$ and $q_{2i+2} = COINIT$, for $i < n$. Suppose $Q' \in BASIC^*$ is such that $\mathscr{L}^Q \not\leqslant \mathscr{L}^{Q'}$. Then, $SEMI\_HULL^{n,S} \not\leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q'}$.*

8. Q-classes which are reducible to $SEMI\_HULL^{n,S}$

In this section, we establish the best possible lower bound on the complexity of $SEMI\_HULL^{n,S}$ in terms of the Q-classes. One can ask the question: using a learner powerful enough to learn $SEMI\_HULL^{n,S}$, can a learner learn languages from the hierarchy based on $BASIC$? The next result shows that, using a learner able to learn $SEMI\_HULL^{n,S}$, one can learn all languages in $(HALF, INIT, \ldots, INIT)$, where $INIT$ is taken $n-1$ times.

We begin with two useful technical propositions.

Proposition 19. *Suppose S is* $\mathbf{rat}^+$*-covering. Then there exists an $S' \subseteq S$, such that S' is* $\mathbf{rat}^+$*-covering, and for all $b, b' \in S'$, if $b < b'$ then $2b < b'$.*

Proof. Suppose h is an isomorphism from Z to S such that $h(i) < h(i+1)$. Define h' as follows: $h'(0) = h(0)$. Suppose we have defined $h'(i)$ and $h'(-i)$. Then define $h'(i+1)$ and $h'(-i-1)$ as follows. $h'(i+1) = h(j)$, such that $h(j) > 2h'(i)$, and $h'(-i-1) = h(j')$ such that $h(j') < h'(-i)/2$. Note that such j and j' exist since S is $\mathbf{rat}^+$-covering. It immediately follows that $S' = \text{range}(h')$ satisfies the requirements of the proposition. $\square$

Proposition 20. *Suppose $S' \subseteq S$. Then $SEMI\_HULL^{n,S'} \leqslant^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.*

Proof. Follows trivially, since $SEMI\_HULL^{n,S'} \subseteq SEMI\_HULL^{n,S}$. $\square$

Theorem 5. *Suppose S is $\mathbf{rat}^+$-covering. Let $n \in N^+$, and $Q = (q_1, q_2, \ldots, q_n)$, where $q_1 = HALF$, and for $2 \leqslant i \leqslant n$, $q_i = INIT$.*

Then, $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Proof. The intuitive idea of the reduction is as follows. Fix the first break point to be $a_1 = 1$. The first $HALF$-component can be reduced to the first slope b_1. Then the $(j-1)$th $INIT$-component i_j is reduced to a_j. Then b_j is fixed based on a_j in such a way that $i_j < i'_j$ would imply $a_j < a'_j$ and $b_j > b'_j + \sum_{j < i \leqslant n} b'_i$, for any potential values of b'_i, $i > j$. This ensures the desired monotonicity in the definition of the reducing operator Θ.

Now we proceed with the formal proof.

Without loss of generality (using Propositions 19 and 20) we can assume that

(Property**H**) for each $b, b' \in S$, if $b < b'$, then $2b < b'$.

Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

For $i_1 \in Z$, and $i_2, \ldots, i_n \in N$, let $map(i_1, i_2, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n)$, where a_j, b_j, $1 \leqslant j \leqslant n$ are defined as follows: $a_1 = 1$, $b_1 = h(i_1 - a_1)$. Suppose we have defined $a_1, b_1, \ldots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1 \leqslant i \leqslant k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1}+1)$th least element in A_{k+1}. Let $b_{k+1} = h(i_1 - a_{k+1})$.

Claim 3. *$(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$ implies $SEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)} \subset SEMI\_HULL^{n,S}_{map(i'_1,\ldots,i'_n)}$.*

Proof. Suppose $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$. Suppose $map(i_1, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n)$ and $map(i'_1, \ldots, i'_n) = (a'_1, b'_1, \ldots, a'_n, b'_n)$.

We consider the following cases.

Case 1: $i_1 > i'_1$.

In this case,

(1) $b_1 > b'_1$, and

(2) $a_1 = a'_1 = 1$.

From (1) and (Property**H**) it follows that

(3) $b_1 > 2 * b'_1$.

Now, for $1 \leqslant i < n$, since $a'_i < a'_{i+1}$, and $b'_i = h(i'_1 - a'_i)$, we have $b'_i > b'_{i+1}$. Thus, $b'_i > 2 * b'_{i+1}$, for $1 \leqslant i < n$ by hypothesis about elements of S.

Thus, $\sum_{1 \leqslant i \leqslant n} b'_i \leqslant 2b'_1$. Along with (3), we have that $b_1 > \sum_{1 \leqslant i \leqslant n} b'_i$. This along with (2), Propositions 11 and 15 gives $SEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subseteq SEMI\_HULL^{1,S}_{a_1,b_1} \subset SEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Case 2: For some j, $1 < j \leqslant n$, for $1 \leqslant k < j$, $i_k = i'_k$, but $i_j < i'_j$.

In this case,

(4) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leqslant i < j$.

(5) $b_j > b'_j$, and

(6) $a_j < a'_j$.

From (5) and (Property**H**) it follows that

(7) $b_j > 2 * b'_j$.

Now, for $1 \leqslant i < n$, since $a'_i < a'_{i+1}$, and $b'_i = h(i'_1 - a'_i)$, we have $b'_i > b'_{i+1}$. Thus, $b'_i > 2 * b'_{i+1}$, for $1 \leqslant i < n$ by hypothesis about elements of S.

Thus, $\sum_{j \leqslant i \leqslant n} b'_i \leqslant 2b'_j$. This, along with (7) gives us that $b_j > \sum_{j \leqslant i \leqslant n} b'_i$. Thus using (6), Propositions 11 and 15 we have $SEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subseteq SEMI\_HULL^{j,S}_{a_1,b_1,\ldots,a_j,b_j} \subset SEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Claim follows from above cases. $\square$

For any $X \subseteq N$, let $\Theta(X) = \bigcup_{\langle i_1,\ldots,i_n \rangle \in X} SEMI\_HULL^{n}_{map(i_1,\ldots,i_n)}$.

Thus, it follows that $\Theta(L^{Q}_{i_1,i_2,\ldots,i_n}) = SEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)}$.

Define Ψ as follows. If a sequence α of grammar converges to a grammar for $SEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)}$, then $\Psi(\alpha)$ converges to a grammar for $L^{Q}_{i_1,i_2,\ldots,i_n}$.

It is now easy to verify that Θ and Ψ witness that $\mathscr{L}^{Q} \leqslant^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$. $\square$

We now show that the above result is in some sense the best possible with respect to the Q-classes considered in this paper. For example, as Corollaries 9 and 10 show, being able to learn the classes $SEMI\_HULL^{n,S}$ cannot help to learn languages even in the classes $(COINIT, COINIT)$ and $(INIT, COINIT)$.

The following technical result will be used to show that the Q-classes with $n+1$ $INIT$s in Q cannot be reduced to $SEMI\_HULL^{n,S}$.

Theorem 6. *Let $n \in N^+$. Suppose $0 \leqslant j \leqslant n$. $Q = (q_1, q_2, \ldots, q_{n+1-j})$, where $q_i = INIT$, for $1 \leqslant i \leqslant n-j+1$. Let $R = R_1 \times R_2 \times \cdots \times R_{n+1-j}$, where R_1 is of cardinality at least 2 and, for $2 \leqslant i \leqslant n+1-j$, R_i is an infinite subset of N. Suppose S is* **rat**$^+$*-covering, and $(a_1, b_1, \ldots, a_j, b_j)$ is S-valid.*

Then

$$\mathscr{L}^{Q,R} \not\leqslant^{\mathbf{TxtEx}} \{SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} \mid (a_1, b_1, \ldots, a_j, b_j, a'_{j+1}, b'_{j+1}, \ldots, a'_n, b'_n) \text{ is } S\text{-valid}\}.$$

Proof. We prove the theorem by reverse induction on j (from n to 0). For $j = n$, the theorem clearly holds (since $\mathscr{L}^{Q,R}$ contains at least two languages, whereas $\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} \mid (a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n)$ is S-valid$\}$ contains only one language).

So suppose the theorem holds for $j' < j \leqslant n$. Then, we show that the theorem holds for $j = j'$.

Suppose by way of contradiction that Θ (along with some Ψ) witnesses that $\mathscr{L}^{Q,R} \leqslant^{\mathbf{TxtEx}} \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} \mid (a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n)$ *is* S-valid$\}$, for some S-valid $(a_1,b_1,\ldots,a_j,b_j)$.

Let w be the minimal element in R_1. For $i \in R_2$, suppose $\Theta(L^Q_{w,i,\min(R_3),\min(R_4),\ldots}) = SEMI\_HULL^{n,b}_{a_1,b_1,\ldots,a_j,b_j,a_{j+1}(i),b_{j+1}(i),\ldots,a_n(i),b_n(i)}$. If, as i varies, $a_{j+1}(i)$ takes arbitrarily large value then, for $w' > w$, $w' \in R_1$, $\Theta(L^Q_{w',\min(R_2),\min(R_3),\ldots}) = SEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j}$ and is thus not a member of $\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} | (a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n)$ is S-valid$\}$.

So suppose $i_1 \in R_2$ maximizes $a_{j+1}(i_1)$. Now for $i > i_1$, if $b_{j+1}(i)$ takes arbitrarily small value, then for $w' > w$, $\Theta(L^Q_{w',\min(R_2),\min(R_3),\ldots}) \supseteq \bigcup_{b'_{j+1} \in S} SEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j,a_{j+1}(i_1),b'_{j+1}}$. However, any member of $\{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} | (a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n)$ is S-valid$\}$, misses out infinitely many elements in $\bigcup_{b'_{j+1} \in S} SEMI\_HULL^j_{a_1,b_1,\ldots,a_j,b_j,a_{j+1}(i_1),b'_{j+1}}$. Thus, $\Theta(L^Q_{w',\min(R_2),\min(R_3),\ldots}) \notin \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n} | (a_1,b_1,\ldots,a_j,b_j,a'_{j+1},b'_{j+1},\ldots,a'_n,b'_n)$ is S-valid$\}$.

So, suppose $i_2 > i_1$, $i_2 \in R_2$, minimizes $b_1(i_2)$.

Thus, Θ (along with Ψ) essentially witnesses that $\mathscr{L}^{Q',R'} \leqslant^{\mathbf{TxtEx}} \{SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,a_{j+1}(i_2),b_{j+1}(i_2),a'_{j+2},b'_{j+2},\ldots,a'_n,b'_n} \mid (a_1,b_1,\ldots,a_j,b_j, \quad a_{j+1}(i_2),b_{j+1}(i_2), a'_{j+2},b'_{j+2},\ldots,a'_n,b'_n)$ is S-valid$\}$, where Q' is obtained by dropping q_1 from Q. R' is obtained from R by dropping R_1 and changing R_2 to $R_2 - \{x | x \leqslant i_2\}$.

This is a contradiction to induction hypothesis. $\square$

Corollary 7. *Suppose* $Q = (q_1,q_2,\ldots,q_{n+1})$, *where* $q_i = INIT$, *for* $1 \leqslant i \leqslant n+1$. *Suppose* S *is* $\mathbf{rat}^+$*-covering.*

Then, $\mathscr{L}^Q \nleqslant^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Proof similar to the one used in proving Theorem 6 above can be used to show the following theorem (we just need to interchange the role of w and w' in the proof, for $j = 0$ case).

Theorem 7. *Let* $n \in N^+$, *and* $Q = (q_1,q_2,\ldots,q_{n+1})$, *where* $q_1 = COINIT$, *and* $q_i = INIT$, *for* $2 \leqslant i \leqslant n+1$. *Let* $R = R_1 \times R_2 \times \cdots \times R_{n+1}$, *where* R_1 *is of cardinality at least* 2 *and, for* $2 \leqslant i \leqslant n+1$, R_i *is an infinite subset of* N. *Suppose* S *is any* $\mathbf{rat}^+$*-covering set.*

Then, $\mathscr{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Corollary 8. *Suppose* $Q = (q_1, q_2, \ldots, q_{n+1})$, *where* $q_1 = COINIT$, *and* $q_i = INIT$, *for* $2 \leqslant i \leqslant n+1$. *Suppose* S *is* $\mathbf{rat}^+$*-covering.*

Then, $\mathscr{L}^{Q} \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Do there exist Q-classes reducible to $SEMI\_HULL^{n,S}$ with $COINIT$ on the second or greater positions in Q? Our next results show that even $(COINIT, COINIT)$ and $(INIT, COINIT)$ are not reducible to $SEMI\_HULL^{n,S}$.

Based on Corollary 3, let us define $<_{\text{valid}}$ as follows.

Definition 19. Suppose $(a_1, b_1, \ldots, a_n, b_n)$ and $(a'_1, b'_1, \ldots, a'_n, b'_n)$ are valid. Then, $(a_1, b_1, \ldots, a_n, b_n) <_{\text{valid}} (a'_1, b'_1, \ldots, a'_n, b'_n)$ iff there exists an i, $1 \leqslant i \leqslant n$ such that, for $1 \leqslant j < i$, $a_j = a'_j$, $b_j = b'_j$ and $a_i < a'_i$ or $a_i = a'_i$ and $b_i > b'_i$.

Note that $<_{\text{valid}}$ imposes a total order among valid sequences of form $(a_1, b_1, \ldots, a_n, b_n)$. Moreover, $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subset SEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$, implies $(a_1, b_1, \ldots, a_n, b_n) <_{\text{valid}} (a'_1, b'_1, \ldots, a'_n, b'_n)$.

Similarly define $(a_1, b_1, \ldots, a_n, b_n) \leqslant_{\text{valid}} (a'_1, b'_1, \ldots, a'_n, b'_n)$ iff $(a_1, b_1, \ldots, a_n, b_n) <_{\text{valid}} (a'_1, b'_1, \ldots, a'_n, b'_n)$ or $(a_1, b_1, \ldots, a_n, b_n) = (a'_1, b'_1, \ldots, a'_n, b'_n)$.

$\geqslant_{\text{valid}}$ and $>_{\text{valid}}$ can be similarly defined.

For any fixed $n \in N$, $X \subseteq \{(a_1, b_1, \ldots, a_n, b_n) \mid (a_1, b_1, \ldots, a_n, b_n) \in VALID_S\}$, let $\max_{\text{valid}}(X) = (a_1, b_1, \ldots, a_n, b_n) \in X$, if any, such that $(\forall (a'_1, b'_1, \ldots, a'_n, b'_n) \in X)[(a'_1, b'_1, \ldots, a'_n, b'_n) \leqslant_{\text{valid}} (a_1, b_1, \ldots, a_n, b_n)]$.

Similarly, let $\min_{\text{valid}}(X) = (a_1, b_1, \ldots, a_n, b_n) \in X$, if any, such that $(\forall (a'_1, b'_1, \ldots, a'_n, b'_n) \in X)[(a_1, b_1, \ldots, a_n, b_n) \leqslant_{\text{valid}} (a'_1, b'_1, \ldots, a'_n, b'_n)]$.

Proposition 21. *Suppose* $n \in N$, S *is* $\mathbf{rat}^+$*-covering, and* $X \subseteq \{(a_1, b_1, \ldots, a_n, b_n) \mid (a_1, b_1, \ldots, a_n, b_n) \in VALID_S\}$. *Then, if* $\min_{\text{valid}}(X)$ *does not exist, then no subset of* $(\bigcap_{I \in X} SEMI\_HULL^n_I)$ *belongs to* $SEMI\_HULL^{n,S}$.

Proof. Suppose S, X is given as above, and $\min_{\text{valid}}(X)$ does not exist. Then, there exists a j, such that for some $a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j$, there exist arbitrarily large b_j such that $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j, \ldots) \in X$, for some $a_{j+1}, b_{j+1}, \ldots, a_n, b_n$, (which may depend on b_j). Thus, $\bigcap_{I \in X} SEMI\_HULL^n_I \subseteq \{(x, y) \mid x \leqslant a_j\}$. But no subset of $\{(x, y) \in N^2 \mid x \leqslant a_j\}$ belongs to $SEMI\_HULL^{n,S}$ (any language L in $SEMI\_HULL^{n,S}$ satisfies: for all x, there exists a y such that $(x, y) \in L$). $\square$

Theorem 8. *Suppose* S *is* $\mathbf{rat}^+$*-covering and* $n \in N$. *Let* $Q = (COINIT, COINIT)$, $R = R_1 \times R_2$, *where* R_2 *is infinite, and* R_1 *contains at least two elements. Then* $\mathscr{L}^{Q,R} \not\leq^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Proof. Suppose by way of contradiction that $\mathscr{L}^{Q,R} \leqslant^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$ as witnessed by Θ (along with Ψ).

Let $w = \min(R_1)$. Suppose $\Theta(L^Q_{w,x}) = SEMI\_HULL^n_{a_1(x),b_1(x),\ldots a_n(x),b_n(x)}$, for $x \in R_2$.

Note that, for $x < x'$, $x, x' \in R_2$, $\Theta(L^Q_{w,x}) \supset \Theta(L^Q_{w,x'})$. Thus, for $x < x'$, $x, x' \in R_2$, $(a_1(x), \ldots, b_n(x)) >_{\text{valid}} (a_1(x'), \ldots, b_n(x'))$, by Corollary 3.

Let $X = \{(a_1(x), b_1(x), \ldots, a_n(x), b_n(x)) | x \in R_2\}$.

If $\min_{\text{valid}}(X)$ does not exist, then for any $w_1 > w$, $w_1 \in R_1$ and $w_2 \in R_2$, $\Theta(L^Q_{w_1,w_2}) \subseteq \bigcap_{x \in R_2} SEMI\_HULL^n_{a_1(x),b_1(x),\ldots}$, does not belong to $SEMI\_HULL^{n,S}$ by Proposition 21.

So suppose $x_1 \in R_2$ is such that $\min_{\text{valid}}(X) = (a_1(x_1), b_1(x_1), \ldots, a_n(x_1), b_n(x_1))$. But then, for all $x \geqslant x_1$, $(a_1(x), \ldots, b_n(x)) \leqslant_{\text{valid}} (a_1(x_1), \ldots, b_n(x_1))$ (due to monotonicity of Θ). Thus, for all $x \geqslant x_1$, $x \in R_2$, $(a_1(x), \ldots, b_n(x)) = (a_1(x_1), \ldots, b_n(x_1))$. A contradiction to Θ (along with Ψ) witnessing that $\mathscr{L}^{Q,R} \leqslant^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$. $\square$

Corollary 9. *Suppose S is* **rat**$^+$*-covering, and $n \in N$. Let $Q = (COINIT, COINIT)$. Then* $\mathscr{L}^Q \nleqslant^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Similar to Theorem 8 we also have the following theorem (for proving it, we just need to interchange the roles of $L^Q_{w,x}$ and $L^Q_{w_1,x}$ in the proof of Theorem 8).

Theorem 9. *Suppose $n \in N$ and S is* **rat**$^+$*-covering. Let $Q = (INIT, COINIT)$, $R = R_1 \times R_2$, where R_2 is infinite, and R_1 contains at least two elements. Then* $\mathscr{L}^{Q,R} \nleqslant^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

Corollary 10. *Suppose S is* **rat**$^+$*-covering. Let $Q = (INIT, COINIT)$. Then* $\mathscr{L}^Q \nleqslant^{\mathbf{TxtEx}} SEMI\_HULL^{n,S}$.

9. Definitions for complements of open semi-hull

In this section, we define the classes of complements of *SEMI_HULL*s and establish some useful propositions following from appropriate definitions.

Definition 20. Suppose $a_1, \ldots, a_n \in N$ and $b_1, \ldots, b_n \in \mathbf{rat}^+$, where $0 < a_1 < a_2 < \cdots < a_n$.

$$\begin{aligned} coSEMI\_HULL^n_{a_1,b_1,a_2,b_2,\ldots,a_n,b_n} &= \left\{ (x,y) \in N^2 | y < \sum_{1 \leqslant i \leqslant n} b_i * (x \dot{-} a_i) \right\} \\ &= N^2 - SEMI\_HULL^n_{a_1,b_1,a_2,b_2,\ldots,a_n,b_n}. \end{aligned}$$

Definition 21. Suppose $S \subseteq \mathbf{rat}^+$ is **rat**$^+$-*covering*.

$$coSEMI\_HULL^{n,S} = \{coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} | (a_1, b_1, \ldots, a_n, b_n) \in VALID_S\}.$$

Proposition 22. *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$ *and* $(a'_1, b'_1, \ldots, a'_m, b'_m)$ *are valid.*
Then, $SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subseteq SEMI\_HULL^m_{a'_1,b'_1,\ldots,a'_m,b'_m}$ *iff* $coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \supseteq coSEMI\_HULL^m_{a'_1,b'_1,\ldots,a'_m,b'_m}$.

Proof. Follows from the definitions. □

Definition 22. Suppose $a_1, b_1, \ldots, , a_j, b_j$ are given such that $(a_1, b_1, \ldots, a_j, b_j) \in VALID$. Then, let $coINTER(a_1, b_1, \ldots, a_j, b_j) =$
$\bigcup\{coSEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n} \mid n \geqslant j \wedge (a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \in VALID\} =$
$N^2 - INTER(a_1, b_1, \ldots, a_j, b_j)$.

Proposition 23. *Suppose* $1 \leqslant j \leqslant n$, *and* $(a_1, b_1, \ldots, \ldots, a_n, b_n)$ *and* $(a'_1, b'_1, \ldots, a'_m, b'_m)$ *are valid. Then* $coINTER(a'_1, b'_1, \ldots, a'_m, b'_m) \supseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ *iff* $INTER(a'_1, b'_1, \ldots, a'_m, b'_m) \subseteq SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$.

Proof. Follows from the definitions. □

Proposition 24. *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1})$ *is valid. Let* $(x, y) \in N^2$ *be such that* $y > \sum_{1 \leqslant i < j} b_i(x \dot{-} a_i)$. *Then, there exists a* $B' \in \mathbf{rat}^+$ *obtainable effectively from* $a_1, b_1, \ldots, a_{j-1}, b_{j-1}, x$ *and* y *such that for any* (a'_j, b'_j), *if* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, *is valid and* $(x, y) \in coINTER^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}$, *then* $a'_j < x$ *and* $b'_j \geqslant B'$.

Proof. Follows from Proposition 16. □

10. Classes to which $coSEMI\_HULL^{n,S}$ is reducible

In this section, we obtain nearly the best possible upper bound on the complexity of $coSEMI_HULL$s in terms of Q-degrees. Intuitive upper bound $\mathscr{L}^Q$ for $Q = (q_1, q_2, \ldots, q_{2n-1}, q_{2n})$ with $q_{2i+1} = COINIT$ and $q_{2i+2} = HALF$ can be easily established using the following learning strategy for $coSEMI\_HULL^{n,S}$: apply a $COINIT$-type strategy to learn the first break point a_1, then apply a $HALF$-type strategy to learn the first slope b_1, etc. However, the upper bound established below contains only $n+1$ components!

Theorem 10. *Suppose* $n \in N$ *and* S *is* $\mathbf{rat}^+$*-covering. Suppose* $Q = (q_1, q_2, \ldots, q_{n+1})$, *where* $q_1 = INIT$ *and* $q_i = COINIT$, *for* $2 \leqslant i \leqslant n+1$. *Then* $coSEMI\_HULL^{n,S} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^Q$.

Proof. The intuitive strategy for learning $coSEMI\_HULL^{n,S}$ providing the desired upper bound operates as follows: first, it applies an $INIT$-type strategy to learn a bound on the maximum slope: $\sum_i b_i$; then, using this bound, it applies *COINIT*-type strategies to learn every pair of parameters a_i, b_i. This *COINIT*-type strategy is in some sense mirror image of the *INIT*-type strategy used in the proof of Theorem 3. The technical details though become somewhat more complicated.

Now we proceed with the formal proof.

Fix S which is $\mathbf{rat}^+$-covering. Let h be a recursive bijection from Z to S such that $h(i) < h(i+1)$. Fix code as in Proposition 18.

Claim 4. *Suppose* $B \in S$. *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j), (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \in VALID_S$, *where* $b_i, b'_i \leqslant B$, *and* $(a_j, b_j) \neq (a'_j, b'_j)$. *Suppose* $coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$.

Then, $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$.

Proof. By Proposition 14 and definitions of $SEMI\_HULL, coSEMI\_HULL, INTER, coINTER$, we have that $a'_j \leqslant a_j$.

Now, since, $coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$, we have $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$. Thus, we have $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) < code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$ (by Proposition 18; for getting $<$ instead of $\leqslant$ use the fact that $(a_j, b_j) \neq (a'_j, b'_j)$). $\square$

Let *map* be a mapping from *VALID* to N^* such that

$$map(a_1, b_1, \ldots, a_n, b_n) = (h^{-1}(B), code(B, a_1, b_1), code(B, a_1, b_1, a_2, b_2), \ldots, code(B, a_1, b_1, \ldots, a_n, b_n)),$$

where B is the least element of S such that $\max(h(0), \sum_{1 \leqslant i \leqslant n} b_i) \leqslant B$.

Claim 5. *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$ *is S-valid. Let* $B = \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1 \leqslant i \leqslant n} b_i)\})$.

(A) *Suppose* $1 \leqslant j \leqslant n$. *Suppose* $B \geqslant b'_j$, *and* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ *is S-valid. If* $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) > code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$. *Then* $coINTER(a_1, b_1, \ldots, a'_j, b'_j) \not\supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$.

(B) *Suppose* $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *is S-valid, and* $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$. *Then either* $B > \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1 \leqslant i \leqslant n} b'_i)\})$ *or for the least* j, $1 \leqslant j \leqslant n$, *such that* $(a_j, b_j) \neq (a'_j, b'_j)$, $coINTER(a'_1, b'_1, \ldots, a'_j, b'_j) \not\supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$.

(C) *Suppose* $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *is S-valid, and* $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$. *Then either* $B > \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1 \leqslant i \leqslant n} b'_i)\})$ *or* $coINTER(a'_1, b'_1, \ldots, a'_j, b'_j, \ldots, a'_n, b'_n) \not\supseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$.

(D) *If* $coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \subset coSEMI\_HULL^n_{a'_1,b'_1,\ldots,a'_n,b'_n}$, *then* $map(a_1, b_1, \ldots, a_n, b_n) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n)$.

(E) *Suppose* $1 \leqslant j \leqslant n$. *There exists a finite* $X_j \subseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j}$ *such that, for all S-valid* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j)$, *if* $B \geqslant b''_j$ *and* $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j) > code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$, *then* $coINTER(a_1, b_1, \ldots, a''_j, b''_j) \not\supseteq X_j$.

(F) *There exists a finite* $X \subseteq coSEMI\_HULL^n_{a_1,b_1,\dots,a_n,b_n}$ *such that, for all S-valid* $(a'_1,b'_1,\dots,a'_n,b'_n)$, *if* $B = \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b'_i)\})$, *and* $map(a'_1,b'_1,\dots,a'_n,b'_n) <_Q map(a_1,b_1,\dots,a_n,b_n)$, *then* $coINTER(a'_1,b'_1,\dots,a'_n,b'_n) \not\supseteq X$.

Proof. (A) Follows from Claim 4.

(B) Let $B' = \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b'_i)\})$. Since $map(a'_1,b'_1,\dots,a'_n,b'_n) <_Q map(a_1,b_1,\dots,a_n,b_n)$, $h^{-1}(B') \leqslant h^{-1}(B)$. Therefore, $B' \leqslant B$, and we must have $\min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b'_i)\}) \leqslant \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b_i)\}) = B$. If $\min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b'_i)\}) = B$, then let j be the least value such that $1 \leqslant j \leqslant n$, and $(a'_j,b'_j) \neq (a_j,b_j)$. Thus, $code(B,a_1,b_1,\dots,a_{j-1},b_{j-1},a'_j,b'_j) > code(B,a_1,b_1,\dots,a_{j-1},b_{j-1},a_j,b_j)$. Now part (B) follows from part (A).

(C) Since $coINTER(a'_1,b'_1,\dots,a'_j,b'_j) \supseteq coINTER(a'_1,b'_1,\dots,a'_j,b'_j,\dots,a'_n,b'_n)$ (by Proposition 12 and definitions of *INTER* and *coINTER*), and $coSEMI\_HULL^j_{a_1,b_1,\dots,a_{j-1},b_{j-1},a_j,b_j} \subseteq coSEMI\_HULL^n_{a_1,b_1,\dots,a_n,b_n}$ (Proposition 11 and definitions of *SEMI_HULL*, *coSEMI_HULL*), part (C) follows from part (B).

(D) Suppose $coSEMI\_HULL^n_{a_1,b_1,\dots,a_n,b_n} \subset coSEMI\_HULL^n_{a'_1,b'_1,\dots,a'_n,b'_n}$. Thus, we must have $\min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b'_i)\}) \geqslant \min(\{b \in S \mid b \geqslant (h(0), \sum_{1\leqslant i\leqslant n} b_i)\})$ (since otherwise, for all but finitely many x, $\sum_{1\leqslant i\leqslant n} b_i(x \dot{-} a_i) > 1 + \sum_{1\leqslant i\leqslant n} b'_i(x \dot{-} a'_i)$, which would contradict $coSEMI\_HULL^n_{a_1,b_1,\dots,a_n,b_n} \subseteq coSEMI\_HULL^n_{a'_1,b'_1,\dots,a'_n,b'_n}$).

If $\min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b'_i)\}) > \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b_i)\})$, then clearly, $map(a_1,b_1,\dots,a_n,b_n) <_Q map(a'_1,b'_1,\dots,a'_n,b'_n)$. If $\min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b'_i)\}) = \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b_i)\})$, then (D) follows from part (C), and the fact that $coSEMI\_HULL^n_{a'_1,b'_1,\dots,a'_n,b'_n} \subseteq coINTER(a'_1,b'_1,\dots,a'_n,b'_n)$.

(E) Let $(x,y) \in N^2$ be such that $(x,y) \in coSEMI\_HULL^n_{a_1,b_1,\dots,a_j,b_j} \subseteq coSEMI\_HULL^n_{a_1,b_1,\dots,a_n,b_n}$, but $y > \sum_{1\leqslant i<j} b_i(x \dot{-} a_i)$. Note that there exists such (x,y).

By Proposition 24, there exists a $B' \in \mathbf{rat}^+$ such that, if $(x,y) \in coINTER^j_{a_1,b_1,\dots,a_{j-1},b_{j-1},a''_j,b''_j}$, then $a''_j < x$ and $b'_j \geqslant B'$. Thus, it follows that for any (a''_j,b''_j), if $a''_j \geqslant x$, or $b''_j < B'$, $(x,y) \notin coINTER(a_1,b_1,\dots,a_{j-1},b_{j-1},a''_j,b''_j)$. Now, for each $a''_j < x$, and $b''_j \in S$ such that $B' \leqslant b''_j \leqslant B$, if $coSEMI\_HULL^j_{a_1,b_1,\dots,a_j,b_j} \not\subseteq coINTER(a_1,b_1,\dots,a_{j-1},b_{j-1},a''_j,b''_j)$, then pick $x_{a''_j,b''_j}, y_{a''_j,b''_j}$ such that $(x_{a''_j,b''_j}, y_{a''_j,b''_j}) \in coSEMI\_HULL^j_{a_1,b_1,\dots,a_j,b_j} - coINTER(a_1,b_1,\dots,a_{j-1},b_{j-1},a''_j,b''_j)$; if $coSEMI\_HULL^j_{a_1,b_1,\dots,a_j,b_j} \subseteq coINTER(a_1,b_1,\dots,a_{j-1},b_{j-1},a''_j,b''_j)$, then let $(x_{a''_j,b''_j}, y_{a''_j,b''_j}) = (x,y)$.

Now let $X_j = \{(x,y)\} \cup \{(x_{a''_j,b''_j}, y_{a''_j,b''_j}) \mid a''_j < x, B' \leqslant b''_j \leqslant B, b''_j \in S\}$. Using part (A), it is easy to verify that X_j witnesses the claim of part (E).

(F) Let $X = \bigcup_{1\leqslant j\leqslant n} X_j$ where X_j is as in part (E). Now, if $map(a'_1,b'_1,\dots,a'_n,b'_n) <_Q map(a_1,b_1,\dots,a_n,b_n)$, and $B = \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1\leqslant i\leqslant n} b'_i)\})$, then there exists a j, $1 \leqslant j \leqslant n$ such for $1 \leqslant i < j$, $a_i = a'_i$ and $b_i = b'_i$ and

$code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) > code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$. Part (F) now follows from part (E) and definition of X. $\square$

We now continue with the proof of the theorem. The aim is to construct Θ which maps $coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ to $L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$ to a grammar sequence converging to a grammar for $coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ would be trivial. We thus just define Θ.

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $Prop(X, a_1, b_1, \ldots, a_n, b_n)$ be true iff following three properties are satisfied. Let $B = \min(\{b \in S \mid b \geqslant h(0) \wedge (\forall (x, y) \in X)[b \geqslant \frac{y}{x}]\})$.

(A) $(a_1, b_1, \ldots, a_n, b_n) \in VALID_S$,

(B) $B \geqslant \min(\{b \in S \mid b \geq \max(h(0), \sum_{1 \leqslant i \leqslant n} b_i)\})$, and

(C) If $B = \min(\{b \in S \mid b \geqslant \max(h(0), \sum_{1 \leqslant i \leqslant n} b_i)\})$, then for all j, $1 \leqslant j \leqslant n$, for all $a'_j \in N, b'_j \in S$ such that $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ is S-valid, $B \geqslant b'_j$ and $code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) > code(B, a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j))$, $[X \not\subseteq coINTER_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j}]$.

Note that whether $X, a_1, b_1, \ldots, a_n, b_n$, satisfy (A) and (B) and (C) is effectively testable.

Moreover, (C) along with definition of B above implies that

(D) For all $(a'_1, b'_1, \ldots, a'_n, b'_n) \in VALID$ such that $b'_i \in S$ and $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, $X \not\subseteq coSEMI\_HULL(a'_1, b'_1, \ldots, a'_n, b'_n)$.

Thus, for finite $X \subseteq N^2$, let

$$\Theta(X) = \bigcup \{L^Q_{map(a_1,b_1,\ldots,a_n,b_n)} \mid Prop(X, a_1, b_1, \ldots, a_n, b_n)\}.$$

For infinite X', $\Theta(X') = \bigcup_{X \subseteq X',\, card(X) < \infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$,

$$\Theta(X) \subseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$$

(by (D) and the fact that for any valid I and I', $map(I) <_Q map(I')$, implies $L^Q_{map(I)} \subseteq L^Q_{map(I')}$), and

(2) for all S-valid $(a_1, b_1, \ldots, a_n, b_n)$, by Claim 5(F), there exists a finite $X \subseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$, such that $\Theta(X) \supseteq L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$.

Thus, we have that $\Theta(coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}) = L^Q_{map(a_1,b_1,\ldots,a_n,b_n)}$. $\square$

The construction in the proof of above theorem can be slightly changed along the following lines: instead of learning first the bound on the maximum slope, one can first apply a *COINIT*-type strategy trying to learn the parameters a_1, b_1 under assumption that b_1 is smaller than some fixed bound $B \in \mathbf{rat}^+$, and then apply an *INIT*-type strategy to learn both the bound on the maximum slope and b_1 if the latter becomes greater than the bound B. Thus, we obtain the following theorem.

Theorem 11. *Suppose* $n \in N$ and S is $\mathbf{rat}^+$*-covering. Suppose* $Q = (q_1, q_2, \ldots, q_{n+1})$, *where* $q_2 = INIT$ and $q_i = COINIT$, *for* $1 \leqslant i \leqslant n+1$, $i \neq 2$. *Then* $coSEMI\_HULL^{n,S} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^Q$.

11. Classes which are reducible to $coSEMI\_HULL^{n,S}$

In this section, we will get a lower bound for $coSEMI\_HULL^{n,S}$ having n components, and thus being very close to the upper bounds obtained in the previous section.

The proof of the following theorem is similar to the proof of the lower bound for $SEMI_HULL$s, with $COINIT$s replacing $INIT$s.

Theorem 12. *Suppose* S *is* $\mathbf{rat}^+$*-covering. Let* $n \in N^+$, *and* $Q = (q_1, q_2, \ldots, q_n)$, *where* $q_1 = HALF$, *and for* $2 \leqslant i \leqslant n$, $q_i = COINIT$.

Then, $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} coSEMI\_HULL^{n,S}$.

The proof of the above theorem is given in Appendix A.

Note that upper and lower bounds for $coSEMI\_HULL^{n,S}$ given by Theorems 10–12 do not match. The lower bound in Theorem 12 above is the best possible (for Q-classes involving components from $BASIC$). However, it is open whether the upper bound can be improved for general n. For $n = 1$, we do know that the upper bound can be improved to show that $coSEMI\_HULL^{1,S} \leqslant^{\mathbf{TxtEx}} HALF$ (which is optimal by Theorem 12).

12. Open hulls—intersections of semi-hulls

Now consider the class of language-figures that are intersections of $SEMI_HULL$s adjacent to the x-axis (that is with the first break point $(a_1, 0)$) and reverse $SEMI_HULL$s adjacent to the y-axis (with the first break point $(0, a'_1)$). These figures are the open hulls.

We give the formal definition below (preceded by the formal definition of the reverse $SEMI_HULL$s adjacent to the y-axis).

Definition 23. $REV\_SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} = \{(x,y) \mid (y,x) \in SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}\}$.

$REV\_SEMI\_HULL^{n,S} =$
$\{REV\_SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \mid SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \in SEMI\_HULL^{n,S}\}$.

Definition 24.
$OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} = SEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n} \cap REV\_SEMI\_HULL^m_{c_1,d_1,\ldots,c_m,d_m}$.

$OP\_HULL^{n,m,S} = \{OP\_HULL^{n,m,S}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} \mid SEMI\_HULL^{n}_{a_1,b_1,\ldots,a_n,\ b_n} \in SEMI\_HULL^{n,S}$, $REV\_SEMI\_HULL^{n}_{c_1,d_1,\ldots,c_m,d_m} \in REV\_SEMI\_HULL^{m,S}$, and $\sum_{1\leqslant i\leqslant n} b_i < \frac{1}{\sum_{1\leqslant i\leqslant m} d_i}\}$.

The latter condition, $\sum_{1\leqslant i\leqslant n} b_i < \frac{1}{\sum_{1\leqslant i\leqslant m} d_i}$, ensures that the languages in *OP_HULL*s are infinite, and thus the corresponding geometrical figures are *open* hulls.[2]

Surprisingly, unlike *SEMI_HULL*s and *coSEMI_HULL*, upper and lower bounds for *OP_HULL*s match. The following theorem establishes the lower bound for the *OP_HULL*s. Somewhat surprising is also the fact that the learnability degree of open hulls is below the learnability degree of semi-hulls. However, the reader must note that while slopes of segments of the border line for semi-hulls are bounded only by both axes, the slopes of the segments of border lines in a open hull bound each other (the reader should note that, as we only consider (reversed) semi-hulls with at least one angle in the theorems below, the (reversed) semi-hull cannot contain the entire domain). Consequently, while learning the angles along both border lines in a open hull, the learning algorithm becomes aware of these bounds and, as it turns out, can use a shorter sequence of the primitive strategies to learn the concept in question.

Theorem 13. *Suppose S is* $\mathbf{rat}^+$*-covering. Suppose $n\geqslant 1$, $m\geqslant 1$. Let $Q=(q_1,\ldots,q_n)$, where each $q_i = INIT$. Then,* (a) $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} OP\_HULL^{n,m,S}$, *and* (b) $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} OP\_HULL^{m,n,S}$.

The proof of the above theorem is given in Appendix A. We next show the upper bound for *OP_HULL*s.

Theorem 14. *Suppose S is* $\mathbf{rat}^+$*-covering. Suppose $n\geqslant m\geqslant 1$. Let $Q=(q_1,\ldots,q_n)$, where each $q_i = INIT$. Then,* (a) $OP\_HULL^{n,m,S} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^Q$, *and* (b) $OP\_HULL^{m,n,S} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^Q$.

The proof of the above theorem is given in Appendix A.

13. Complements of open hulls

In this section, we define and explore the classes of complements of *OP_HULL*s.

Definition 25. $REV\_coSEMI\_HULL^{n}_{a_1,b_1,\ldots,a_n,b_n} = \{(x,y) \mid (y,x) \in coSEMI\_HULL^{n}_{a_1,b_1,\ldots,a_n,b_n}\}$.
$REV\_coSEMI\_HULL^{n,S}$
$= \{REV\_coSEMI\_HULL^{n}_{a_1,b_1,\ldots,a_n,b_n} \mid coSEMI\_HULL^{n}_{a_1,b_1,\ldots,a_n,b_n} \in coSEMI\_HULL^{n,S}\}$.

[2] If we do not require $\sum_{1\leqslant i\leqslant n} b_i < \frac{1}{\sum_{1\leqslant i\leqslant m} d_i}$, then the geometrical figure may be finite. In this case, the complexity of learnability shows similar properties as the classes considered (with the above constraint on slopes), however the analysis becomes more complex.

Definition 26. $coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} = coSEMI\_HULL^{n}_{a_1,b_1,\ldots,a_n,b_n} \cup REV\_\ coSEMI\_\ HULL^{m}_{c_1,d_1,\ldots,c_m,d_m} = N^2 - OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$.

$coOP\_HULL^{n,m,S} = \{coOP\_HULL^{n,m,S}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} \mid coSEMI\_HULL^{n}_{a_1,b_1,\ldots,a_n,b_n} \in coSEMI\_HULL^{n,S}$, $REV\_coSEMI\_HULL^{m}_{c_1,d_1,\ldots,c_m,d_m} \in REV\_coSEMI\_HULL^{m,S}$, and $\sum_{1\leqslant i\leqslant n} b_i < \frac{1}{\sum_{1\leqslant i\leqslant m} d_i}\}$.

The following theorem gives the lower bound for *coOP_HULL*s.

Theorem 15. *Suppose S is* **rat**$^+$*-covering. Suppose $n\geqslant 1$, $m\geqslant 1$. Let $Q=(q_1,\ldots,q_n)$, where each $q_i = COINIT$. Then* (a) $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} coOP\_HULL^{n,m,S}$ *and* (b) $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} coOP\_HULL^{m,n,S}$.

Proof of the above theorem is given in Appendix A. The following theorem gives the upper bound for *coOP_HULL*s.

Theorem 16. *Suppose $n\geqslant m\geqslant 1$. Let $Q=(q_1,\ldots,q_n)$, where each $q_i = COINIT$. Then* (a) $coOP\_HULL^{n,m,S} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^Q$ *and* (b) $coOP\_HULL^{m,n,S} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^Q$.

Proof of the above theorem is given in Appendix A.

14. Conclusions

A new complexity scale has been successfully applied for evaluating the complexity of learning various geometrical figures from texts. Many upper bounds obtained by us are surprisingly lower than the ones suggested by intuitive learning strategies. Another surprising result is that upper and lower bounds match for *OP_HULL* and their complements, while there is a gap between upper and lower bounds for *SEMI_HULL*s that cannot be narrowed. One more interesting aspect of this picture is that upper bounds for *OP_HULL*s, the intersection of *SEMI_HULL*s, are much lower than that for *SEMI_HULL*s themselves! In general, the picture of upper and lower bounds for *OP_HULL*s and their complements is much more uniform than for *SEMI_HULL*s and their complements: bounds for *coOP_HULL*s can be obtained from the bounds for *OP_HULL*s by just replacing *INIT*s by *COINIT*s, while bounds for *SEMI_HULL*s and *coSEMI_HULL*s differ even in the number of components in Q-vectors.

There are many other interesting types of geometrical concepts whose complexity can be explored in terms of the Q-classes. For example, one can evaluate the complexity of learning *SEMI_HULL*s and all other figures observed in our paper dropping requirement of the first angle being adjacent to x- or y-axis. Even more promising seems to be the class of finite unions of *OP_HULL*s (though proofs may become technically messy). In general, we are convinced that the Q-classes (possibly using some other basic classes/strategies) are very promising tools for exploring the complexity of learning hard languages from texts.

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Appendix A

A.1. Proof of Theorem 1

Theorem 1. $HALF \equiv^{\mathbf{TxtEx}} INIT \times COINIT$.

Proof. For $a \in Z$, let $L_a = \{x \in Z \mid x \geqslant a\}$. For $i,j \in N$, let $X_{i,j} = \{x \in N \mid x \leqslant i\} \times \{x \mid x \geqslant j\}$.

We now define Θ and Ψ witnessing that $HALF \leqslant^{\mathbf{TxtEx}} INIT \times COINIT$.

For any finite subset Y of Z, let $\Theta(Y) = \bigcup_{a \geqslant 0, a \in Y} X_{0,a} \cup \bigcup_{a<0, a \in Y} X_{-a,0}$.

It can be easily verified that $\Theta(L_a) = X_{0,a}$ (if $a \geqslant 0$), and $\Theta(L_a) = X_{-a,0}$ (if $a<0$).

Ψ is defined as follows. Suppose a sequence α of grammars converges to grammar p. Suppose $i = \max(\{x \in N \mid (\exists y \in N)[\langle x,y \rangle \in W_p]\})$, and $j = \min(\{y \in N \mid (\exists x \in N)[\langle x,y \rangle \in W_p]\})$. Then, if $i=0$, then $\Psi(\alpha)$ converges to a program for L_j. If $i>0$, then $\Psi(\alpha)$ converges to a program for L_{-i}.

It is easy to verify that Θ and Ψ witness that $HALF \leqslant^{\mathbf{TxtEx}} INIT \times COINIT$.

Now, we show that $INIT \times COINIT \leqslant^{\mathbf{TxtEx}} HALF$.

Define h as follows:

for $i,j \in N$,

$$h(i,j) = -[(i+1)(i+2)/2] + 1 + j, \text{ if } i \geqslant j;$$
$$h(i,j) = [j(j+1)/2] - i, \text{ if } i<j.$$

Intuitively, picture $h(i,j)$ as follows:

$\cdots$	$h(2,0)$	$h(2,1)$	$h(2,2)$	$h(1,0)$	$h(1,1)$	$h(0,0)$
$\cdots$	-5	-4	-3	-2	-1	0
$h(0,1)$	$h(1,2)$	$h(0,2)$	$h(2,3)$	$h(1,3)$	$h(0,3)$	$\cdots$
1	2	3	4	5	6	$\cdots$

Note that for all $i,j,k,l \in N$, if $i \geqslant k$ and $j \leqslant l$, then $h(i,j) \leqslant h(k,l)$. Now, for any set $Y \subseteq N^2$, let $\Theta(Y) = \bigcup_{\langle i,j \rangle \in Y} L_{h(i,j)}$. It is easy to verify that $\Theta(X_{i,j}) = L_{h(i,j)}$.

$\Psi(\alpha)$ is defined as follows. If a sequence α of grammars converges to a grammar p for $L_{h(i,j)}$, then $\Psi(\alpha)$ converges to a grammar for $X_{i,j}$. (Note that, if p, is a grammar for some $L_{h(i,j)}$, then such

i,j can be determined in the limit from p.) It is easy to verify that Θ and Ψ witness that $INIT \times COINIT \leqslant^{\mathbf{TxtEx}} HALF$. $\square$

A.2. Proof of Theorem 2

Theorem 2. *Suppose $Q = (q_1, \ldots, q_k) \in BASIC^k$ and $Q' = (q'_1, \ldots, q'_l) \in BASIC^l$. Let $R = R_1 \times R_2 \times \cdots \times R_k$, $R' = R'_1 \times R'_2 \times \cdots \times R'_l$, where each R_i (R'_i) is an infinite subset of N, if $q_i \in \{INIT, COINIT\}$ ($q'_i \in \{INIT, COINIT\}$), and R_i (R'_i) is a subset of Z, with infinite intersection with both N and Z^-, if $q_i = HALF$ ($q'_i = HALF$).*

If Q is not a pseudo-subsequence of Q' then $\mathscr{L}^{Q,R} \not\leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q',R'}$.

Proof. We prove the theorem by double induction (first on k and then on l). For $k = 0$ or $l = 0$ theorem clearly holds. Suppose by induction that the theorem holds for $k \leqslant m$, $l \in N$, and for $k = m+1$, $l \leqslant r$. We then show that the theorem holds for $k = m+1$ and $l = r+1$. Suppose by way of contradiction that Θ (along with Ψ) witnesses that $\mathscr{L}^{Q,R} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q',R'}$.

We consider the following cases:

Case 1: $q_1 = INIT$.

Case 1.1: $q'_1 = COINIT$.

Consider σ, which minimizes $i \in N$ such that $\langle i, \ldots \rangle \in content(\Theta(\sigma))$. Let $j \in N$ be the maximum number such that $\langle j, \ldots \rangle \in content(\sigma)$. It follows that, for any $j' > j$, $j' \in R_1$, $\Theta(L^Q_{j',\ldots})$, for any values of other parameters, is of the form $L^{Q'}_{i,\ldots}$, for some values of the other parameters. Thus, Θ (along with Ψ) essentially witnesses that $\mathscr{L}^{Q,RR} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leqslant j\}$, and QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1. Now we are done by induction hypothesis.

Case 1.2: $q'_1 = INIT$.

In this case, $k \geqslant 2$.

Case 1.2.1: $q_2 = INIT$.

Fix $i_1 \in R_1$, and consider the set $\bigcup_{i_2 \in R_2,\ldots} \Theta(L^Q_{i_1,i_2,\ldots})$. Suppose this set contains $\langle i'_1, \ldots \rangle$, for arbitrarily large i'_1. Then for any $ii_1 > i_1$ (since $L^Q_{ii_1,\ldots} \supseteq L^Q_{i_1,\ldots}$ for all possible values of other parameters) we have that $\Theta(L^Q_{ii_1,\min(R_2),\min(R_3),\ldots})$ contains elements of form $\langle i'_1, \ldots \rangle$ for arbitrarily large i'_1. Thus, $\Theta(L^Q_{ii_1,\min(R_2),\min(R_3),\ldots}) \notin \mathscr{L}^{Q'}$.

So let i'_1 be maximum value such that some element of form $\langle i'_1, \ldots \rangle$ is in $\bigcup_{i_2 \in R_2,\ldots} \Theta(L^Q_{i_1,i_2,\ldots})$.

Let σ be such that $content(\sigma) \subseteq L^Q_{i_1,i_2,\ldots}$, and $\Theta(\sigma)$ contains an element of form $\langle i'_1, \ldots \rangle$. Let i_2 be maximum value such that some element of form $\langle i_1, i_2, \ldots \rangle$ is in $content(\sigma)$. It follows that, for all $ii_2 > i_2$, $ii_2 \in R_2$, $\Theta(L^Q_{i_1,ii_2,\ldots})$, for any value of other parameters, is of form $L^{Q'}_{i'_1,\ldots}$, for some value of other parameters. Thus, Θ (along with Ψ) essentially witnesses that $\mathscr{L}^{QQ,RR} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^{QQ',RR'}$, where QQ is obtained from Q by dropping q_1, QQ' is obtained from Q' by dropping q'_1, RR' is obtained from R' by dropping R'_1 and RR is obtained from R by dropping R_1 plus changing R_2 to $R_2 - \{x \mid x \leqslant i_2\}$. Now we are done by induction hypothesis.

Case 1.2.2: $q_2 = COINIT$.

In this case, by Proposition 10(a) QQ obtained from Q by dropping q_1 is not a pseudo-subsequence of Q'. Thus, we are done by induction hypothesis.

Case 1.2.3: $q_2 = HALF$.

If $l = 1$, then we are done. So assume $l \geqslant 2$.

Case 1.2.3.1: $q'_2 = INIT$ or $HALF$.

Then, by replacing q_2 by $COINIT$, using Proposition 10(d) we still have that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 1.2.2.

Case 1.2.3.2: $q'_2 = COINIT$.

Then, by replacing q_2 by $INIT$, using Proposition 10(e) we still have that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 1.2.1.

Case 1.3: $q'_1 = HALF$.

In this case, let $i \in R_1$. Suppose $\Theta(L^Q_{i,\ldots}) = L^{Q'}_{j,\ldots}$, for some values of other parameters. But then for all $i' > i$, $\Theta(L^Q_{i',\ldots})$, for any value of other parameters, must be of form $L^{Q'}_{j',\ldots}$, for some value of other parameters, where $j' \leqslant j$. Thus, one could essentially consider Θ (along with Ψ) as a reduction from $\mathscr{L}^{Q,RR}$ to $\mathscr{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leqslant i\}$, QQ' is obtained from Q' by replacing q'_1 with $INIT$, and RR' is obtained from R' by replacing R'_1 by $\{x \in N \mid -x + j \in R'_1 - \{y \mid y \geqslant j\}\}$. Thus, we can use Case 1.2.

Case 2: $q_1 = COINIT$. This case is very similar to Case 1. We give the analysis for completeness sake.

Case 2.1: $q'_1 = INIT$.

Let $i \in N$ be minimum value such that $\Theta(L^Q_{\ldots}) = L^{Q'}_{i,\ldots}$, for some values of the other parameters. Let $j \in R_1$ be such that $\Theta(L^Q_{j,\ldots}) = L^{Q'}_{i,\ldots}$, for some values of the parameters. It follows that for all $j' > j$, $j' \in R_1$, $\Theta(L^Q_{j',\ldots})$, for any value of other parameters, is of form $L^{Q'}_{i,\ldots}$, for some value of other parameters.

Thus, Θ (along with Ψ) essentially witnesses that $\mathscr{L}^{Q,RR} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^{QQ',RR'}$, where RR is obtained from R by replacing R_1 by $R_1 - \{x \mid x \leqslant j\}$, and QQ' is obtained from Q' by dropping q'_1 and RR' is obtained from R' by dropping R'_1. Now we are done by induction hypothesis.

Case 2.2: $q'_1 = COINIT$.

In this case $k \geqslant 2$.

Case 2.2.1: $q_2 = INIT$.

In this case, by Proposition 10(a) QQ obtained from Q by dropping q_1 is not a pseudo-subsequence of Q'. Thus, we are done by induction hypothesis.

Case 2.2.2: $q_2 = COINIT$. Fix $i_1 \in R_1$, and consider $\Theta(L^Q_{i_1,\ldots}) = L^{Q'}_{i'_1,\ldots}$. If i'_1 achieves arbitrary high value (for some values of other parameters) then, for $ii_1 > i_1$, since $L^Q_{ii_1,0,0,\ldots} \subseteq L^Q_{i_1,\ldots}$, and $\bigcap_{i'_1 \in R'_1} L^{Q'}_{i'_1,\ldots} = \emptyset$, $\Theta(L_{ii_1,0,0,\ldots}) = \emptyset \notin \mathscr{L}^{Q',R'}$.

So let i_1' be maximum value such that for some value of other parameters, $\Theta(L^{Q,R}_{i_1,\dots}) = L^{Q',R'}_{i_1',\dots}$. Let i_2 be such that, for some value of other parameters, $\Theta(L^{Q,R}_{i_1,i_2,\dots}) = L^{Q',R'}_{i_1',\dots}$. It follows that, for all $ii_2 > i_2$, $\Theta(L^{Q,R}_{i_1,ii_2,\dots})$ is of form $L^{Q',R'}_{i_1',\dots}$. Thus, Θ (along with Ψ) essentially witnesses that $\mathscr{L}^{QQ,RR} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^{QQ',RR'}$, where QQ is obtained from Q by dropping q_1, QQ' is obtained from Q' by dropping q_1', RR' is obtained from R' by dropping R_1' and RR is obtained from R by dropping R_1 plus changing R_2 to $R_2 - \{x \mid x \leqslant i_2\}$. Now we are done by induction hypothesis.

Case 2.2.3: $q_2 = HALF$.

If $l = 1$, then we are done. So assume $l \geqslant 2$.

Case 2.2.3.1: $q_2' = INIT$ or $HALF$.

Then, by replacing q_2 by $COINIT$, using Proposition 10(d), we still have that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 2.2.2.

Case 2.2.3.2: $q_2' = COINIT$.

Then, by replacing q_2 by $INIT$, using Proposition 10(e), we still have that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 2.2.1.

Case 3: $q_1 = HALF$.

Case 3.1: $q_1' = INIT$.

Then by Proposition 10(b) replacing q_1 by $COINIT$, still gives us that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 2.

Case 3.2: $q_1' = COINIT$.

Then by Proposition 10(c) replacing q_1 by $INIT$, still gives us that Q is not a pseudo-subsequence of Q'. Thus, we can use Case 1.

Case 3.3: $q_1' = HALF$.

In this case, let $i \in R_1$ and consider $\Theta(L^{Q}_{i,\dots})$, for some value of other parameters. Suppose it is $L^{Q'}_{j,\dots}$, for some value of other parameters. Then for all $i' < i$, $i' \in R_1$, $\Theta(L^{Q}_{i',\dots})$ must be of form $L^{Q'}_{j',\dots}$, where $j' \leqslant j$. Thus, one could essentially consider this as a reduction from $\mathscr{L}^{QQ,RR}$ to $\mathscr{L}^{QQ',RR'}$, where QQ is obtained from Q by replacing q_1 by $INIT$, RR is obtained from R by replacing R_1 by $\{x \in N \mid -x + i \in R_1 - \{y \mid y \geqslant i\}\}$, QQ' is obtained from Q' by replacing q_1' with $INIT$, and RR' is obtained from R' by replacing R_1' by $\{x \in N \mid -x + j \in R_1' - \{y \mid y \geqslant j\}\}$. Now we can use Case 1.

It follows from above cases that $\mathscr{L}^{Q,R} \not\leqslant^{\mathbf{TxtEx}} \mathscr{L}^{Q',R'}$. $\square$

A.3. Proof of Theorem 12

Proposition A.1. *Suppose $S' \subseteq S$. Then $coSEMI\_HULL^{n,S'} \leqslant^{\mathbf{TxtEx}} coSEMI\_HULL^{n,S}$.*

Theorem 12. *Suppose S is* $\mathbf{rat}^+$*-covering. Let $n \in N^+$, and $Q = (q_1, q_2, \dots, q_n)$, where $q_1 = HALF$, and for $2 \leqslant i \leqslant n$, $q_i = COINIT$.*

Then, $\mathscr{L}^{Q} \leqslant^{\mathbf{TxtEx}} coSEMI\_HULL^{n,S}$.

Proof. Without loss of generality (using Propositions 19 and A.1) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

Let $map(i_1, i_2, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n)$, where a_j, b_j, $1 \leqslant j \leqslant n$ are defined as follows. $a_1 = 1$, $b_1 = h(-i_1 - a_1)$. Suppose we have defined $a_1, b_1, \ldots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1 \leqslant i \leqslant k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1}+1)$th least element in A_{k+1}. Let $b_{k+1} = h(-i_1 - a_{k+1})$.

Claim A.1. $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$ *implies* $coSEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)} \subset coSEMI\_HULL^{n,S}_{map(i'_1,\ldots,i'_n)}$.

Proof. Suppose $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$. Suppose $map(i_1, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n)$ and $map(i'_1, \ldots, i'_n) = (a'_1, b'_1, \ldots, a'_n, b'_n)$.

We consider the following cases.

Case 1: $i_1 > i'_1$.

In this case,

(1) $b_1 < b'_1$, and

(2) $a_1 = a'_1$.

From (1) it follows that

(3) $2b_1 < b'_1$.

Now, for $1 \leqslant i < n$, since $a_i < a_{i+1}$, and $b_i = h(-i_1 - a_i)$, we have $b_i > b_{i+1}$. Thus, $b_i > 2 * b_{i+1}$, for $1 \leqslant i < n$ by hypothesis about elements of S. Thus, $\sum_{1 \leqslant i \leqslant n} b_i \leqslant 2b_1$. Along with (3), we have that $b'_1 \geqslant \sum_{1 \leqslant i \leqslant n} b_i$. This along with (2), Propositions 11, 15 and 22 gives $coSEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subseteq coSEMI\_HULL^{1,S}_{a'_1,b'_1} \subseteq coSEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

Case 2: For some j, $1 < j \leqslant n$, $i_k = i'_k$, for $1 \leqslant k < j$, but $i_j > i'_j$.

In this case

(4) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leqslant i < j$.

(5) $b_j < b'_j$, and

(6) $a_j > a'_j$.

From (5) it follows that

(7) $2 * b_j < b'_j$.

Now, for $1 \leqslant i < n$, since $a_i < a_{i+1}$, $b_i = h(-i_1 - a_i)$, we have $b_i > b_{i+1}$. Thus, $b_i > 2 * b_{i+1}$, for $1 \leqslant i < n$, by hypothesis about elements of S.

Thus, $\sum_{j \leqslant i \leqslant n} b_i \leqslant 2b_j$. This, along with (7) gives us that $b'_j > \sum_{j \leqslant i \leqslant n} b_i$. Thus using (6), Propositions 11, 15, and 22 we have $coSEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subseteq coSEMI\_HULL^{j,S}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j} \subseteq coSEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

The claim follows from the above cases. □

We let $\Theta(X) = \bigcup_{\langle i_1,\ldots,i_n \rangle \in X} coSEMI\_HULL^{n}_{map(i_1,\ldots,i_n)}$.

Thus, it follows that $\Theta(L^Q_{i_1,i_2,\ldots,i_n}) = coSEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)}$.

Define Ψ as follows. If a sequence α of grammars converges to a grammar for $coSEMI\_HULL^{n,S}_{map(i_1,\ldots,i_n)}$, then $\Psi(\alpha)$ converges to a grammar for $L^Q_{i_1,i_2,\ldots,i_n}$.

It is now easy to verify that Θ and Ψ witness that $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} coSEMI\_HULL^{n,S}$. $\square$

A.4. Proofs of Theorems 13 and 14

The following proposition is an obvious corollary of the definition of OP_HULLs.

Proposition A.2. *Suppose $S \subseteq S'$. Then $OP\_HULL^{n,m,S} \leqslant^{\mathbf{TxtEx}} OP\_HULL^{n,m,S'}$.*

Theorem 13. *Suppose S is* **rat**$^+$*-covering. Suppose $n \geqslant 1$, $m \geqslant 1$. Let $Q = (q_1, \ldots, q_n)$, where each $q_i = INIT$. Then,* (a) $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} OP\_HULL^{n,m,S}$, *and* (b) $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} OP\_HULL^{m,n,S}$.

Proof. We only show part (a). Part (b) can be proved similarly.

The desired reduction works very similarly to the analogous reduction in Theorem 5: we just fix a REV_SEMI_HULL and try to reduce a language in $\mathscr{L}^Q$ to the $SEMI\_HULL$ part of the OP_HULL; the slope of the REV_SEMI_HULL provides a starting point for learning the first slope b_1, thus $HALF$ being replaced with $INIT$ in the first component of Q.

Now we proceed with the formal proof.

Without loss of generality (using Propositions 19 and A.2) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

Let $(c_1, d_1, \ldots, c_m, d_m)$ be S-valid such that $\sum_{1 \leqslant i \leqslant m} d_i < 1/h(1)$ (note that there clearly exist such $c_1, d_1, \ldots, c_m, d_m$).

Let $map(i_1, i_2, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$, where a_j, b_j, $1 \leqslant j \leqslant n$ are defined as follows. $a_1 = i_1 + 1$, $b_1 = h(-a_1)$. Suppose we have defined $a_1, b_1, \ldots, a_k, b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k, [\sum_{1 \leqslant i \leqslant k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1}+1)$th least element in A_{k+1}. Let $b_{k+1} = h(-a_{k+1})$.

Note that, for all $(i_1, i_2, \ldots, i_n) \in N^n$, if $map(i_1, i_2, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$, then since $a_i < a_{i+1}$, by definition of b_i, we have $b_i = h(-a_i) > h(-a_{i+1}) = b_{i+1}$. This along with requirement on S gives $b_i > 2b_{i+1}$. Thus, $\sum_{1 \leqslant i \leqslant n} b_i \leqslant 2b_1 \leqslant 2h(-a_1) \leqslant 2h(0) < h(1)$. Since, $\sum_{1 \leqslant i \leqslant m} d_i < 1/h(1)$, we immediately have that $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}$, for all $(i_1, i_2, \ldots, i_n)$.

Claim A.2. $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$ *implies* $OP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)} \subset OP\_HULL^{n,m,S}_{map(i'_1,\ldots,i'_n)}$.

Proof. Suppose $(i_1, i_2, \ldots, i_n) <_Q (i'_1, i'_2, \ldots, i'_n)$. Suppose $map(i_1, \ldots, i_n) = (a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$ and $map(i'_1, \ldots, i'_n) = (a'_1, b'_1, \ldots, a'_n, b'_n; c_1, d_1, \ldots, c_m, d_m)$.

Let j be the least number such that $i_k = i'_k$, for $1 \leqslant k < j$, and $i_j \neq i'_j$.

Thus,

(1) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leqslant i < j$.

(2) $b_j > b'_j$, and

(3) $a_j < a'_j$.

From (2) it follows that

(4) $b_j > 2 * b'_j$.

Now, for $1 \leqslant i < n$, since $a'_i < a'_{i+1}$, we have $b'_i = h(-a'_i) > h(-a'_{i+1}) = b'_{i+1}$. Thus, $b'_i > 2 * b'_{i+1}$, for $1 \leqslant i < n$ by hypothesis about elements of S.

Thus, $\sum_{j \leqslant i \leqslant n} b'_i \leqslant 2b'_j$. This, along with (4) gives us that $b_j > \sum_{j \leqslant i \leqslant n} b'_i$. Thus using (3), Propositions 11 and 15 we have $SEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subset SEMI\_HULL^{j,S}_{a_1,b_1,\ldots,a_j,b_j} \subseteq SEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

The claim follows. $\square$

We now define $\Theta(X)$ for any finite set X as follows.

We let $\Theta(X) = \bigcup_{\langle i_1,\ldots,i_n \rangle \in X} OP\_HULL^{n,m}_{map(i_1,\ldots,i_n)}$.

Thus, it follows that $\Theta(L^Q_{i_1,i_2,\ldots,i_n}) = OP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)}$.

Define Ψ as follows. If a sequence α of grammars converges to a grammar for $OP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)}$, then $\Psi(\alpha)$ converges to a grammar for $L^Q_{i_1,i_2,\ldots,i_n}$.

It is now easy to verify that Θ and Ψ witness that $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} OP\_HULL^{n,m,S}$. $\square$

Before proving Theorem 14 we need some propositions.

Definition A.1. Suppose $(a_1, b_1, \ldots, a_j, b_j)$ is valid. $REV\_INTER(a_1, b_1, \ldots, a_j, b_j) = \{(x, y) \mid (y, x) \in INTER(a_1, b_1, \ldots, a_j, b_j)\}$.

Definition A.2. Suppose $(a_1, b_1, \ldots, a_j, b_j)$ and $(c_1, d_1, \ldots, c_k, d_k)$ are valid. Then, $INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k) =$
$\cap\{OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n;c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m} \mid n \geqslant j$ and $m \geqslant k$ and $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$ are valid and $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}\}$.

Proposition A.3. *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$ *and* $(c_1, d_1, \ldots, c_k, d_k)$ *are valid, and* $\sum_{1 \leqslant i \leqslant j} b_j < \frac{1}{\sum_{1 \leqslant i \leqslant k} d_i}$. *Then,*

(a) $\{(x, y) \mid x \leqslant maxinter(a_1, b_1, \ldots, a_j, b_j), y = \min(\{y' \mid (x, y') \in INTER(a_1, b_1, \ldots, a_j, b_j)\})\} \subseteq INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$.

(b) $\{(x, y) \mid y \leqslant maxinter(c_1, d_1, \ldots, c_k, d_k),\ x = \min(\{x' \mid (x', y) \in REV\_INTER(c_1, d_1, \ldots, c_j, d_j)\})\} \subseteq INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$.

Proof. We show only part (a). Part (b) can be proved similarly. Suppose $(x_0, y_0) \in \{(x, y) \mid x \leqslant maxinter(a_1, b_1, \ldots, a_j, b_j),\ y = \min(\{y' \mid (x, y') \in INTER(a_1, b_1, \ldots, a_j, b_j)\})\}$. Then, $1 + \sum_{1 \leqslant i \leqslant j} b_i(x_0 \dot{-} a_i) > y_0 \geqslant \sum_{1 \leqslant i \leqslant j} b_i(x_0 \dot{-} a_i)$. Thus, $1 + \sum_{1 \leqslant i \leqslant j} b_i * x_0 \geqslant y_0$. Hence,

(1) $\sum_{1 \leqslant i \leqslant j} b_i \geqslant (y_0 - 1)/x_0$.

Clearly, $(x_0, y_0) \in INTER(a_1, b_1, \ldots, a_j, b_j)$. Thus, $(x_0, y_0) \in SEMI\_HULL^n_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n}$, for all valid $(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n)$ (by definition of $INTER$).

Thus, if $(x_0, y_0) \notin INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$, there must exist a valid $(c_1, d_1, \ldots, c_k, d_k, \ldots, c_m, d_m)$, and $\sum_{1 \leqslant i \leqslant j} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}$, such that $(x_0, y_0) \notin REV\_SEMI\_HULL^m_{c_1, d_1, \ldots, c_k, d_k, \ldots, c_m, d_m}$. But this would mean, $x_0 < \sum_{1 \leqslant i \leqslant m} d_i(y_0 \dot{-} c_i) \leqslant \sum_{1 \leqslant i \leqslant m} d_i(y_0 \dot{-} 1)$. Thus,

(2) $\sum_{1 \leqslant i \leqslant m} d_i > x_0/(y_0 - 1)$.

From (1) and (2) we have

$\sum_{1 \leqslant i \leqslant n} b_i > \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}$. A contradiction to the hypothesis. □

Proposition A.4. *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$, $(a'_1, b'_1, \ldots, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_k, d_k)$, $(c'_1, d'_1, \ldots, c'_k, d'_k)$ *are valid. Suppose further that* $\sum_{1 \leqslant i \leqslant j} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant k} d_i}$, *and* $\sum_{1 \leqslant i \leqslant j} b'_i < \frac{1}{\sum_{1 \leqslant i \leqslant k} d'_i}$.

If $INT\_OP\_HULL(a'_1, b'_1, \ldots, a'_j, b'_j; c'_1, d'_1, \ldots, c'_k, d'_k) \subseteq OP\_HULL^{j,k}_{a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k}$, *then,* $INTER(a'_1, b'_1, \ldots, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_j, b_j}$ *and* $REV\_INTER(c'_1, d'_1, \ldots, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^k_{c_1, d_1, \ldots, c_k, d_k}$.

Proof. Suppose $INT\_OP\_HULL(a'_1, b'_1, \ldots, a'_j, b'_j; c'_1, d'_1, \ldots, c'_k, d'_k) \subseteq OP\_HULL^{j,k}_{a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k}$.

Thus (using Proposition A.3) $\{(x, y) \mid x \leqslant maxinter(a'_1, b'_1, \ldots, a'_j, b'_j)$, $y = \min(\{y' \mid (x, y') \in INTER(a'_1, b'_1, \ldots, a'_j, b'_j)\})\} \subseteq INT\_OP\_HULL(a'_1, b'_1, \ldots, a'_j, b'_j; c'_1, d'_1, \ldots, c'_k, d'_k)$ $\subseteq OP\_HULL_{a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k} \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_j, b_j}$. It follows that $INTER(a'_1, b'_1, \ldots, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_j, b_j}$.

Similarly, it can be shown that

$REV\_INTER(c'_1, d'_1, \ldots, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^k_{c_1, d_1, \ldots, c_k, d_k}$. □

Corollary A.1. *Suppose* $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n$. *Let S be any* **rat**$^+$*-covering set.*

Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$ *are S-valid, and* $\sum_{1 \leqslant i \leqslant j} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant k} d_i}$.

Then, there exist only finitely many (a'_j, b'_j, c'_k, d'_k) *such that*

(i) $b'_j + \sum_{1 \leqslant i < j} b_i < \frac{1}{d'_k + \sum_{1 \leqslant i < k} d_k}$, *and*

(ii) $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k)$ *are S-valid, and*

(iii) $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_j, b_j}$, *and* $REV\_INTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^k_{c_1, d_1, \ldots, c_k, d_k}$.

Moreover, canonical index for the finite set of (a'_j, b'_j, c'_k, d'_k) *satisfying above three conditions can be obtained effectively from* $a_1, b_1, \ldots, a_j, b_j, c_1, d_1, \ldots, c_k, d_k$.

Furthermore, for any (a'_j, b'_j, c'_j, d'_j) *satisfying the above three conditions,* $a'_j \leqslant a_j$, $c'_k \leqslant c_k$, *and if* $a'_j = a_j$ *then* $b'_j \geqslant b_j$, *and if* $c'_k = c_k$, *then* $d'_k \geqslant d_k$.

Proof. Suppose $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^{j}_{a_1,b_1,\ldots,a_j,b_j}$, and $REV\_INTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^{k}_{c_1,d_1,\ldots,c_k,d_k}$. By Corollary 5 and Proposition 14 it follows that $a'_j \leqslant a_j$, $c'_k \leqslant c_k$, and there exists only finitely many (a'_j, b'_j) such that $b'_j \leqslant b_j$ and $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^{j}_{a_1,b_1,\ldots,a_j,b_j}$, and only finitely many (c'_k, d'_k) such that $d'_k \leqslant d_k$ and $REV\_INTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^{k}_{c_1,d_1,\ldots,c_k,d_k}$, and these (a'_j, b'_j), (c'_k, d'_k) can be obtained effectively from $(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$. Let $B'_j = \min(\{b_j\} \cup \{b'_j \mid b'_j \leqslant b_j \wedge INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^{j}_{a_1,b_1,\ldots,a_j,b_j}\})$. Let $D'_k = \min(\{d_k\} \cup \{d'_k \mid d'_k \leqslant d_k \wedge REV\_INTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^{k}_{c_1,d_1,\ldots,c_k,d_k}\})$.

Note that clause (i) in the corollary implies that $b'_j < \frac{1}{d'_k}$ and $d'_k < \frac{1}{b'_j}$.

It follows that, for any (a'_j, b'_j, c'_k, d'_k) to satisfy the hypothesis of the corollary we must have $0 \leqslant a'_j \leqslant a_j$, $0 \leqslant c'_k \leqslant c_k$, $B'_j \leqslant b'_j \leqslant 1/D'_k$, and $D'_k \leqslant d'_k \leqslant 1/B'_j$. Thus, there exist only finitely many (a'_j, b'_j, c'_k, d'_k) which can satisfy clauses (i)–(iii) of the corollary. Moreover, since for any (a'_j, b'_j, c'_k, d'_k) it is effectively testable whether clauses (i)–(iii) of the corollary are satisfiable, we can find the canonical index for the set of (a'_j, b'_j, c'_k, d'_k) satisfying the clauses (i)–(iii) of the corollary effectively from $a_1, b_1, \ldots, a_j, b_j, c_1, d_1, \ldots, c_k, d_k$.

Furthermore clause of the corollary follows using Corollary 5. $\square$

Proposition A.5. *Let S be any* **rat**$^+$*-covering set. Then, there exists a recursive function* Icode *with domain $VALID_S \times VALID_S$, and range $\subseteq N$ such that following is satisfied.*

Suppose $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j)$, $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k)$ are S-valid, and $\sum_{1 \leqslant i \leqslant j} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant k} d_i}$ and $b'_j + \sum_{1 \leqslant i < j} b_i < \frac{1}{d'_k + \sum_{1 \leqslant i < k} d_i}$. Then

(A) *If $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^{j}_{a_1,b_1,\ldots,a_j,b_j}$, and $REV\_INTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^{k}_{c_1,d_1,\ldots,c_k,d_k}$ then $Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \leqslant Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$.*

(B) If $(a_j, b_j, c_k, d_k) \neq (a'_j, b'_j, c'_k, d'_k)$, *then $Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \neq Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$.*

(C) $\{Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c''_k, d''_k) \mid (a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j) \in VALID_S, (c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c''_j, d''_j) \in VALID_S,$ *and* $b''_j + \sum_{1 \leqslant i < j} b_j < \frac{1}{d''_k + \sum_{1 \leqslant i < k} d_k}\} = N$.

Proof can be done along similar lines as of Proposition 18.

Now we show the upper bound for OP_HULLs.

Theorem 14. *Suppose S is* **rat**$^+$*-covering. Suppose $n \geqslant m \geqslant 1$. Let $Q = (q_1, \ldots, q_n)$, where each $q_i = INIT$. Then,* (a) $OP\_HULL^{n,m,S} \leqslant^{\mathbf{TxtEx}} \mathcal{L}^Q$, *and* (b) $OP\_HULL^{m,n,S} \leqslant^{\mathbf{TxtEx}} \mathcal{L}^Q$.

Proof. We show only part (a). Part (b) can be done similarly.

Intuitively, we use *INIT*-type strategy to learn every set of parameters (a_i, b_i, c_i, d_i). This is possible based on Proposition A.5 above.

Now we proceed with the formal proof.

Let h be a recursive bijection from Z to S such that $h(i) < h(i+1)$, for $i \in Z$. Let *Icode* be as in Proposition A.5. For $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$ in $VALID_S$, such that $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}$, we let $map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m) = (Icode(a_1, b_1; c_1, d_1), Icode(a_1, b_1, a_2, b_2; c_1, d_1, c_2, d_2), \ldots, Icode(a_1, b_1, \ldots, a_m, b_m; c_1, d_1, \ldots, c_m, d_m), Icode(a_1, b_1, \ldots, a_{m+1}, b_{m+1}; c_1, d_1, \ldots, c_m, d_m), \ldots, Icode(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m))$.

Claim A.3. *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$ *and* $(a'_1, b'_1, \ldots, a'_n, b'_n)$, $(c_1, d_1, \ldots, c_m, d_m)$ *and* $(c'_1, d'_1, \ldots, c'_m, d'_m)$, *are S-valid, and* $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}$, *and* $\sum_{1 \leqslant i \leqslant n} b'_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d'_i}$.

(A) *Suppose* $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$. *Then, for the least* j *such that* $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$, $INTER(a_1, b_1, \ldots, a_j, b_j) \nsubseteq SEMI\_HULL^{j}_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j}$, *or*
$REV\_INTER(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) \nsubseteq REV\_SEMI\_HULL^{j}_{c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}}$.

(B) *Suppose* $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$. *Then,* $INTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n) \nsubseteq SEMI\_HULL^{n}_{a'_1, b'_1, \ldots, a'_n, b'_n}$ *or* $REV\_INTER(c_1, d_1, \ldots, c_m, d_m) \nsubseteq REV\_SEMI\_HULL^{m}_{c'_1, d'_1, \ldots, c'_m, d'_m}$.

(C) *Suppose* $OP\_HULL^{n,m}_{a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m} \subset OP\_HULL^{n,m}_{a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m}$. *Then* $map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m)$.

(D) *Suppose* $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$. *Then,* $\{(x, y) \in N^2 \mid x \leqslant maxinter(a_1, b_1, \ldots, a_n, b_n)$ *and* $y = \min(\{y' \mid (x, y') \in INTER(a_1, b_1, \ldots, a_n, b_n)\})$ *or* $y \leqslant maxinter(c_1, d_1, \ldots, c_m, d_m)$ *and* $x = \min(\{x' \mid (x', y) \in REV\_INTER(c_1, d_1, \ldots, c_m, d_m)\})\} \nsubseteq OP\_HULL^{n,m}_{a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m}$.

Proof. (A) Let j be least number such that $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$. Note that, for $i < j$, we must have $Icode(a_1, b_1, \ldots, a_i, b_i; c_1, d_1, \ldots, c_{\min(i,m)}, d_{\min(i,m)}) = Icode(a'_1, b'_1, \ldots, a'_i, b'_i; c'_1, d'_1, \ldots, c'_{\min(i,m)}, d'_{\min(i,m)})$. If $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq SEMI\_HULL^{j}_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j}$ and $REV\_INTER(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) \subseteq REV\_SEMI\_HULL^{j}_{c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}}$, then, by Proposition A.5 we would have $Icode(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) < Icode(a'_1, b'_1, \ldots, a'_j, b'_j; c'_1, d'_1, \ldots, c'_{\min(j,m)}, d'_{\min(j,m)})$. Thus, $map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_{\min(j,m)}, d'_{\min(j,m)})$, a contradiction to the hypothesis.

(B) Follows from the fact that $INTER(a_1, b_1, \ldots, a_j, b_j) \subseteq INTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n)$, $REV\_INTER(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) \subseteq REV\_INTER(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}, \ldots, c_m, d_m)$,

$SEMI\_HULL^{n}_{a'_1,b'_1,\ldots,a'_n,b'_n} \subseteq SEMI\_HULL^{j}_{a'_1,b'_1,\ldots,a'_j,b'_j}$ and $REV\_SEMI\_HULL^{m}_{c'_1,d'_1,\ldots,c'_m,d'_m} \subseteq REV\_SEMI\_HULL^{m}_{c'_1,d'_1,\ldots,c'_{\min(j,m)},d'_{\min(j,m)}}$ and part (A).

(C) If $OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} \subset OP\_HULL^{n,m}_{a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m}$, then by Proposition A.4, $INTER(a_1,b_1,\ldots,a_n,b_n) \subseteq SEMI\_HULL^{n}_{a'_1,b'_1,\ldots,a'_n,b'_n}$ and $REV\_INTER(c_1,d_1,\ldots,c_m,d_m) \subseteq REV\_SEMI\_HULL^{m}_{c'_1,d'_1,\ldots,c'_m,d'_m}$. Now part (C) follows from part (B).

(D) Follows from (B) and definition of $OP\_HULL^{n,m}$. $\square$

We now continue with the proof of the theorem. The aim is to construct Θ which maps $OP\_HULL^{n,m,S}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ to $L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$ to a grammar sequence converging to a grammar for $OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ would be trivial. We thus just define Θ.

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $Prop(X,a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)$ be true iff following two properties are satisfied.

(A) $(a_1,b_1,\ldots,a_n,b_n) \in VALID_S$, $(c_1,d_1,\ldots,c_m,d_m) \in VALID_S$, and $\sum_{1\leqslant i\leqslant n} b_i < \frac{1}{\sum_{1\leqslant i\leqslant m} d_i}$.

(B) For all $(a'_1,b'_1,\ldots,a'_n,b'_n),(c'_1,d'_1,\ldots,c'_m,d'_m) \in VALID_S$, $\sum_{1\leqslant i\leqslant n} b'_i < \frac{1}{\sum_{1\leqslant i\leqslant m} d'_i}$, such that $map(a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m) <_Q map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)$, $X \not\subseteq OP\_HULL^{n,m}_{a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m}$.

Note that condition (B) above is equivalent to

(B′) For all j, $1\leqslant j\leqslant n$, for all $a'_j, c'_{\min(j,m)} \in N, b'_j, d'_{\min(j,m)} \in S$, such that $b'_j + \sum_{1\leqslant i<j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1\leqslant i<\min(j,m)} d'_i}$,
if $Icode(a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}) < Icode(a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c_{\min(j,m)},d_{\min(j,m)})$, then

$$[X \not\subseteq OP\_HULL^{j,\min(j,m)}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}}].$$

Note that whether $X,a_1,b_1,\ldots,a_n,b_n,c_1,d_1,\ldots,c_m,d_m$, satisfy (A) and (B′), for all j, $1\leqslant j\leqslant n$, is effectively testable.

Thus, for finite $X \subseteq N^2$, let

$$\Theta(X) = \bigcup\{L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n)} \mid Prop(X,a_1,b_1,\ldots,a_n,b_n)\}.$$

For infinite X', $\Theta(X') = \bigcup_{X\subseteq X', card(X)<\infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq OP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$,

$$\Theta(X) \subseteq L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$$

(due to clause (B) in definition of *Prop* above, and the fact that for any valid I and I', $map(I) <_Q map(I')$, implies $L^Q_{map(I)} \subseteq L^Q_{map(I')}$), and

(2) for any finite set X such that $\{(x,y) \in N^2 \mid x \leqslant maxinter(a_1,b_1,\dots,a_n,b_n)$ and $y = \min(\{y' \mid (x,y') \in INTER(a_1,b_1,\dots,a_n,b_n)\})$ or $y \leqslant maxinter(c_1,d_1,\dots,c_m,d_m)$ and $x = \min(\{x' \mid (x',y) \in REV\_INTER(c_1,d_1,\dots,c_m,d_m)\})\} \subseteq X \subseteq OP\_HULL^{n,m}_{a_1,b_1,\dots,a_n,b_n;c_1,d_1,\dots,c_m,d_m}$,

$$\Theta(X) \supseteq L^Q_{map(a_1,b_1,\dots,a_n,b_n;c_1,d_1,\dots,c_m,d_m)}.$$

(By Claim A.3(D), and definition of *Prop* and Θ).

Thus, we have that $\Theta(OP\_HULL^{n,m}_{a_1,b_1,\dots,a_n,b_n;c_1,d_1,\dots,c_m,d_m}) = L^Q_{map(a_1,b_1,\dots,a_n,b_n;c_1,d_1,\dots,c_m,d_m)}$. □

A.5. Proofs of Theorems 15 and 16

Proposition A.6. *Suppose $S \subseteq S'$. Then $coOP\_HULL^{n,m,S} \leqslant^{\mathbf{TxtEx}} coOP\_HULL^{n,m,S'}$.*

The following theorem gives the lower bound for *coOP_HULL*s.

Theorem 15. *Suppose S is $\mathbf{rat}^+$-covering. Suppose $n \geqslant 1$, $m \geqslant 1$. Let $Q = (q_1,\dots,q_n)$, where each $q_i = COINIT$. Then* (a) *$\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} coOP\_HULL^{n,m,S}$ and* (b) *$\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} coOP\_HULL^{m,n,S}$.*

Proof. We only show part (a). Part (b) can be proved similarly. The proof is very similar to the proof of lower bound for *OP_HULL*s with *INIT*s being replaced by *COINIT*s. Without loss of generality (using Propositions 19 and A.6) we can assume that, for each $b, b' \in S$, if $b < b'$, then $2b < b'$. Let h be an isomorphism from Z to S such that $h(i) < h(i+1)$.

Let $(c_1,d_1,\dots,c_m,d_m)$ be S-valid such that $\sum_{1 \leqslant i \leqslant m} d_i < 1/h(1)$ (note that there clearly exist such $c_1,d_1,\dots,c_m,d_m$).

Let $map(i_1,i_2,\dots,i_n) = (a_1,b_1,\dots,a_n,b_n;c_1,d_1,\dots,c_m,d_m)$, where a_j, b_j, $1 \leqslant j \leqslant n$ are defined as follows. $a_1 = i_1 + 1$, $b_1 = h(-a_1)$. Suppose we have defined $a_1,b_1,\dots,a_k,b_k$. Then let $A_{k+1} = \{x \in N \mid x > a_k,\ [\sum_{1 \leqslant i \leqslant k} b_i * (x \dot{-} a_i)] \in N\}$, and then let a_{k+1} to be the $(i_{k+1}+1)$th least element in A_{k+1}. Let $b_{k+1} = h(-a_{k+1})$.

Note that, for all $(i_1,i_2,\dots,i_n) \in N^n$, if $map(i_1,i_2,\dots,i_n) = (a_1,b_1,\dots,a_n,b_n;c_1,d_1,\dots,c_m,d_m)$, then since $a_i < a_{i+1}$, by definition of b_i, we have $b_i = h(-a_i) > h(-a_{i+1}) = b_{i+1}$. This along with requirement on S gives $b_i > 2b_{i+1}$. Thus, $\sum_{1 \leqslant i \leqslant n} b_i \leqslant 2b_1 \leqslant 2h(-a_1) \leqslant 2h(0) < h(1)$. Since, $\sum_{1 \leqslant i \leqslant m} d_i < 1/h(1)$, we immediately have that $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}$, for all $(i_1,i_2,\dots,i_n)$.

Claim A.4. *$(i_1,i_2,\dots,i_n) <_Q (i'_1,i'_2,\dots,i'_n)$ implies $coOP\_HULL^{n,m,S}_{map(i_1,\dots,i_n)} \subset coOP\_HULL^{n,m,S}_{map(i'_1,\dots,i'_n)}$.*

Proof. Suppose $(i_1,i_2,\dots,i_n) <_Q (i'_1,i'_2,\dots,i'_n)$. Suppose $map(i_1,\dots,i_n) = (a_1,b_1,\dots,a_n,b_n;$ $c_1,d_1,\dots,c_m,d_m)$ and $map(i'_1,\dots,i'_n) = (a'_1,b'_1,\dots,a'_n,b'_n;c_1,d_1,\dots,c_m,d_m)$.

Let j be the least number such that $i_k = i'_k$, for $1 \leqslant k < j$, and $i_j \neq i'_j$.

Thus,
(1) $a_i = a'_i$ and $b_i = b'_i$, for $1 \leqslant i < j$.
(2) $b_j < b'_j$, and
(3) $a_j > a'_j$.
From (2) it follows that
(4) $2b_j < b'_j$.

Now, for $1 \leqslant i < n$, since $a_i < a_{i+1}$, we have $b_i = h(-a_i) > h(-a_{i+1}) = b_{i+1}$. Thus, $b_i > 2 * b_{i+1}$, for $1 \leqslant i < n$ by hypothesis about elements of S.

Thus, $\sum_{j \leqslant i \leqslant n} b_i \leqslant 2b_j$. This, along with (4) gives us that $b'_j > \sum_{j \leqslant i \leqslant n} b_i$. Thus using (3), Propositions 11, 15, and 22, we have $coSEMI\_HULL^{n,S}_{a_1,b_1,\ldots,a_n,b_n} \subset coSEMI\_HULL^{j,S}_{a'_1,b'_1,\ldots,a'_j,b'_j} \subseteq coSEMI\_HULL^{n,S}_{a'_1,b'_1,\ldots,a'_n,b'_n}$.

The claim follows. $\square$

We now define $\Theta(X)$ for any finite set X as follows.

We let $\Theta(X) = \cup_{\langle i_1,i_2,\ldots,i_n\rangle \in X} coOP\_HULL^{n,m}_{map(i_1,\ldots,i_n)}$.

Thus, it follows that $\Theta(L^Q_{i_1,i_2,\ldots,i_n}) = coOP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)}$.

Define Ψ as follows. If a sequence α of grammars converges to a grammar for $coOP\_HULL^{n,m,S}_{map(i_1,\ldots,i_n)}$, then $\Psi(\alpha)$ converges to a grammar for $L^Q_{i_1,i_2,\ldots,i_n}$.

It is now easy to verify that Θ and Ψ witness that $\mathscr{L}^Q \leqslant^{\mathbf{TxtEx}} coOP\_HULL^{n,m,S}$. $\square$

Before proving Theorem 16 we need some propositions.

Definition A.3. Suppose $(a_1, b_1, \ldots, a_j, b_j)$ is valid. $REV\_coINTER(a_1, b_1, \ldots, a_j, b_j) = \{(x, y) \mid (y, x) \in co\_INTER(a_1, b_1, \ldots, a_j, b_j)\}$.

Definition A.4. Suppose $(a_1, b_1, \ldots, a_j, b_j)$ and $(c_1, d_1, \ldots, c_k, d_k)$ are valid. Then,
$coINT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$
$= \bigcup\{coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n;c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m} \mid n \geqslant j$ and $m \geqslant k$ and $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$ are valid and $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}\} =$
$N^2 - INT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$.

Proposition A.7. *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$ *and* $(c_1, d_1, \ldots, c_k, d_k)$ *are valid. Suppose* $\sum_{1 \leqslant i \leqslant j} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant k} d_i}$. *Let* $(x, y) \in N^2$ *be such that* $1 + \sum_{1 \leqslant i < j} b_i(x \dot{-} a_i) \geqslant y \geqslant \sum_{1 \leqslant i < j} b_i(x \dot{-} a_i)$. *Then, if* $(x, y) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$, *then* $(x, y) \in coINTER(a_1, b_1, \ldots, a_j, b_j)$.

Proof. Suppose $(x, y) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$. Then, $(x, y) \in \bigcup\{coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_j,b_j,\ldots,a_n,b_n;c_1,d_1,\ldots,c_k,d_k,\ldots,c_m,d_m} \mid n \geqslant j$ and $m \geqslant k$ and $(a_1, b_1, \ldots, a_n, b_n)$ and

$(c_1, d_1, \ldots, c_m, d_m)$ are valid and $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}\}$. Thus,
$(x, y) \in \bigcup\{coSEMI\_HULL^n_{a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n} \mid n \geqslant j$ and $(a_1, b_1, \ldots, a_n, b_n)$ is valid$\} \cup$ $\bigcup\{REV\_coSEMI\_HULL^m_{c_1, d_1, \ldots, c_k, d_k, \ldots, c_m, d_m} \mid m \geqslant k$ and $(c_1, d_1, \ldots, c_m, d_m)$ is valid and $\sum_{1 \leqslant i \leqslant m} d_i < \frac{1}{\sum_{1 \leqslant i < j} b_i}\}$. Thus,
$(x, y) \in coINTER(a_1, b_1, \ldots, a_j, b_j) \cup \bigcup\{REV\_coSEMI\_HULL^m_{c_1, d_1, \ldots, c_k, d_k, \ldots, c_m, d_m} \mid m \geqslant k$ and $(c_1, d_1, \ldots, c_m, d_m)$ is valid and $\sum_{1 \leqslant i \leqslant m} d_i < \frac{1}{\sum_{1 \leqslant i < j} b_i}\}$.

We claim that $(x, y) \notin \bigcup\{REV\_coSEMI\_HULL^m_{c_1, d_1, \ldots, c_k, d_k, \ldots, c_m, d_m} \mid m \geqslant k$ and $(c_1, d_1, \ldots, c_m, d_m)$ is valid and $\sum_{1 \leqslant i \leqslant m} d_i < \frac{1}{\sum_{1 \leqslant i < j} b_i}\}$. This would prove the proposition. Suppose by way of contradiction that $(x, y) \in REV\_coSEMI\_HULL^m_{c_1, d_1, \ldots, c_k, d_k, \ldots, c_m, d_m}$, where $m \geqslant k$, $(c_1, d_1, \ldots, c_m, d_m)$ is valid and $\sum_{1 \leqslant i \leqslant m} d_i < \frac{1}{\sum_{1 \leqslant i < j} b_i}$. But, then $(x, y) \in REV\_coSEMI\_HULL^m_{c_1, \sum_{1 \leqslant i \leqslant m} d_i}$. Thus, $x < \sum_{1 \leqslant i \leqslant m} d_i(y \dot{-} 1)$. If $y \leqslant 1$, then clearly, the above cannot happen. So assume $y > 1$. Thus, $x < \sum_{1 \leqslant i \leqslant m} d_i(y - 1)$. Hence,

(1) $x/(y-1) < \sum_{1 \leqslant i \leqslant m} d_i$.

However, $y \leqslant 1 + \sum_{1 \leqslant i < j} b_i(x \dot{-} a_i)$. Thus, $y - 1 \leqslant \sum_{1 \leqslant i < j} b_i(x \dot{-} a_i)$. Thus, $y - 1 \leqslant \sum_{1 \leqslant i < j} b_i x$. Thus,

(2) $(y-1)/x \leqslant \sum_{1 \leqslant i < j} b_i$.

Multiplying (1) and (2) we have $1 \leqslant [\sum_{1 \leqslant i < j} b_i] * [\sum_{1 \leqslant i \leqslant m} d_i]$. But then, $\sum_{1 \leqslant i \leqslant n} b_i \geqslant \sum_{1 \leqslant i < j} b_i \geqslant 1/[\sum_{1 \leqslant i \leqslant m} d_i]$. A contradiction. □

Similarly, one can show

Proposition A.8. *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$ *and* $(c_1, d_1, \ldots, c_k, d_k)$ *are valid. Suppose* $\sum_{1 \leqslant i \leqslant j} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant k} d_i}$. *Let* $(x, y) \in N^2$ *be such that* $1 + \sum_{1 \leqslant i < j} d_i(y \dot{-} c_i) \geqslant x \geqslant \sum_{1 \leqslant i < j} d_i(y \dot{-} c_i)$. *Then, if* $(x, y) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k)$, *then* $(x, y) \in REV\_coINTER(c_1, d_1, \ldots, c_k, d_k)$.

Proposition A.9. *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$, $(a'_1, b'_1, \ldots, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_k, d_k)$, $(c'_1, d'_1, \ldots, c'_k, d'_k)$, *are valid. Suppose further that* $\sum_{1 \leqslant i \leqslant j} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant k} d_i}$, *and* $\sum_{1 \leqslant i \leqslant j} b'_i < \frac{1}{\sum_{1 \leqslant i \leqslant k} d'_i}$.

If $coINT\_OP\_HULL(a'_1, b'_1, \ldots, a'_j, b'_j; c'_1, d'_1, \ldots, c'_k, d'_k) \supseteq coOP\_HULL^{j,k}_{a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_k, d_k}$, *then* $coINTER(a'_1, b'_1, \ldots, a'_j, b'_j) \supseteq coSEMI\_HULL^j_{a_1, b_1, \ldots, a_j, b_j}$ *and* $REV\_coINTER(c'_1, d'_1, \ldots, c'_k, d'_k) \supseteq REV\_coSEMI\_HULL^k_{c_1, d_1, \ldots, c_k, d_k}$.

Proof. Follows from Proposition A.4. □

Theorem 16. *Suppose* $n \geqslant m \geqslant 1$. *Let* $Q = (q_1, \ldots, q_n)$, *where each* $q_i = COINIT$. *Then* (a) $coOP\_HULL^{n,m,S} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^Q$ *and* (b) $coOP\_HULL^{m,n,S} \leqslant^{\mathbf{TxtEx}} \mathscr{L}^Q$.

Proof. We show only part (a). Part (b) can be done similarly.

Intuitively, we use *COINIT*-type strategy to learn every set of parameters (a_i, b_i, c_i, d_i). This is possible based on Proposition A.5 above using a method similar to that used in Theorem 14, though technical details become more complicated.

Let h be a recursive bijection from Z to S such that $h(i) < h(i+1)$, for $i \in Z$. Let *Icode* be as in Proposition A.5. For $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$ in $VALID_S$, such that $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}$, we let $map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m) = (Icode(a_1, b_1; c_1, d_1),$ $Icode(a_1, b_1, a_2, b_2; c_1, d_1, c_2, d_2), \ldots, Icode(a_1, b_1, \ldots, a_m, b_m; c_1, d_1, \ldots, c_m, d_m),$ $Icode(a_1, b_1, \ldots, a_{m+1}, b_{m+1}; c_1, d_1, \ldots, c_m, d_m), \ldots, Icode(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m))$.

Claim A.5. *Suppose* $(a_1, b_1, \ldots, a_j, b_j)$, $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_k, d_k)$, $(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k)$, *are S-valid, where* $\sum_{1 \leqslant i \leqslant j} b_j < \frac{1}{\sum_{1 \leqslant i \leqslant k} d_k}$, *and* $b'_j + \sum_{1 \leqslant i < j} b_j < \frac{1}{d'_k + \sum_{1 \leqslant i < k} d_k}$. *Suppose* $(a_j, b_j, c_k, d_k) \neq (a'_j, b'_j, c'_k, d'_k)$. *Suppose* $coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ $\supseteq coSEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$ *and* $REV\_coINTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \supseteq$ $REV\_coSEMI\_HULL^k_{c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k}$, *then*
$Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k)$
$< Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$.

Proof. By hypothesis, we have $INTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \subseteq SEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$ and $REV\_INTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \subseteq REV\_SEMI\_HULL^k_{c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k}$. Thus, we have $Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) < Icode(a_1, b_1, \ldots, a_{j-1},$ $b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$ (by Proposition A.5; for getting $<$, use the fact that $(a_j, b_j, c_k, d_k) \neq (a'_j, b'_j, c'_k, d'_k)$). □

Claim A.6. *Suppose* $(a_1, b_1, \ldots, a_n, b_n)$, $(c_1, d_1, \ldots, c_m, d_m)$ *are* S-valid, *and* $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}$.

(A) *Suppose* $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant m$. *Suppose* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$ *and* $(c_1, d_1, \ldots, c_{k-1},$ $d_{k-1}, c'_k, d'_k)$ *are S-valid,* $b'_j + \sum_{1 \leqslant i < j} b_i < \frac{1}{d'_k + \sum_{1 \leqslant i < k} d_i}$.

Suppose $Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) > Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1},$ $a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k)$. *Then*
$[coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \not\supseteq coSEMI\_HULL^j_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j}$
or $REV\_coINTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \not\supseteq REV\_coSEMI\_HULL^k_{c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k}]$. *Thus,* $coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots,$
$c_{k-1}, d_{k-1}, c'_k, d'_k) \not\supseteq coOP\_HULL^{j,k}_{a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c_k, d_k}$.

(B) *Suppose* $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *and* $(c'_1, d'_1, \ldots, c'_m, d'_m)$ *are S-valid,* $\sum_{1 \leqslant i \leqslant n} b'_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d'_i}$. *Suppose* $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$. *Then, for the*

least j such that $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$, $coINTER(a_1, b_1, \ldots, a'_j, b'_j)$ $\not\supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$, *or*
$REV\_coINTER(c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$
$\not\supseteq REV\_coSEMI\_HULL^j_{c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c_{\min(j,m)},d_{\min(j,m)}}$.

(C) *Suppose* $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *and* $(c'_1, d'_1, \ldots, c'_m, d'_m)$ *are S-valid,* $\sum_{1\leqslant i\leqslant n} b'_i < \frac{1}{\sum_{1\leqslant i\leqslant m} d'_i}$. *Suppose* $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$. *Then,* $coINTER(a'_1, b'_1, \ldots, a'_j, b'_j, \ldots, a'_n, b'_n) \not\supseteq coSEMI\_HULL^n_{a_1,b_1,\ldots,a_n,b_n}$ or
$REV\_coINTER(c'_1, d'_1, \ldots, c'_m, d'_m) \not\supseteq REV\_coSEMI\_HULL^m_{c_1,d_1,\ldots,c_m,d_m}$.

(D) *Suppose* $(a'_1, b'_1, \ldots, a'_n, b'_n)$ *and* $(c'_1, d'_1, \ldots, c'_m, d'_m)$ *are S-valid,* $\sum_{1\leqslant i\leqslant n} b'_i < \frac{1}{\sum_{1\leqslant i\leqslant m} d'_i}$. *Suppose* $coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m} \subset coOP\_HULL^{n,m}_{a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m}$. *Then* $map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m)$.

(E) *For* $1\leqslant j\leqslant n$, *there exists a finite* $X_j \subseteq coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ *such that, for all S-valid* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$, *such that* $b'_j + \sum_{1\leqslant i<j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1\leqslant i<\min(j,m)} d_i}$, *if*
$Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$
$> Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$, *then,*
$coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) \not\supseteq X_j$.

(F) *There exists a finite* $X \subseteq coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ *such that, for all* j, $1\leqslant j\leqslant n$, *for S-valid* $(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j)$, $(c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$, *such that* $b'_j + \sum_{1\leqslant i<j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1\leqslant i<\min(j,m)} d_i}$, *if*
$Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$
$> Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$, *then,*
$coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) \not\supseteq X$.

Proof. (A) Suppose the hypothesis. Then, it follows from Claim A.5 that $[coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \not\supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$ or $REV\_coINTER(c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k) \not\supseteq REV\_coSEMI\_HULL^k_{c_1,d_1,\ldots,c_{k-1},d_{k-1},c_k,d_k}]$. Thus, by Proposition A.9, we have $coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{k-1}, d_{k-1}, c'_k, d'_k)$
$\not\supseteq coOP\_HULL^{j,k}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j;c_1,d_1,\ldots,c_{k-1},d_{k-1},c_k,d_k}$.

(B) Let j be least number such that $(a_j, b_j, c_{\min(j,m)}, d_{\min(j,m)}) \neq (a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$. Note that, for $i<j$, we must have $Icode(a_1, b_1, \ldots, a_i, b_i; c_1, d_1, \ldots, c_{\min(i,m)}, d_{\min(i,m)}) = Icode(a'_1, b'_1, \ldots, a'_i, b'_i; c'_1, d'_1, \ldots, c'_{\min(i,m)}, d'_{\min(i,m)})$. If
$coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j) \supseteq coSEMI\_HULL^j_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a_j,b_j}$ and
$REV\_coINTER(c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$
$\supseteq REV\_coSEMI\_HULL^j_{c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c_{\min(j,m)},d_{\min(j,m)}}$, then, by part (A) we would have

$Icode(a_1, b_1, \ldots, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) > Icode(B, a'_1, b'_1, \ldots, a'_j, b'_j; c'_1, d'_1, \ldots, c'_{\min(j,m)}, d'_{\min(j,m)})$. Thus, $map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) <_Q map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_{\min(j,m)}, d'_{\min(j,m)})$, a contradiction to the hypothesis.

(C) Follows from the fact that, for $1 \leqslant j \leqslant n$, $coINTER(a_1, b_1, \ldots, a_j, b_j) \supseteq coINTER(a_1, b_1, \ldots, a_j, b_j, \ldots, a_n, b_n)$, $REV\_coINTER(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}) \supseteq REV\_coINTER(c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}, \ldots, c_m, d_m)$, $coSEMI\_HULL^n_{a'_1, b'_1, \ldots, a'_n, b'_n} \supseteq coSEMI\_HULL^j_{a'_1, b'_1, \ldots, a'_j, b'_j}$ and $REV\_coSEMI\_HULL^m_{c'_1, d'_1, \ldots, c'_m, d'_m} \supseteq REV\_coSEMI\_HULL^j_{c'_1, d'_1, \ldots, c'_{\min(j,m)}, d'_{\min(j,m)}}$ and part (B).

(D) If $coOP\_HULL^{n,m}_{a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m} \subset coOP\_HULL^{n,m}_{a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m}$, then $coOP\_HULL^{n,m}_{a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m} \subset coINT\_OP\_HULL^{n,m}_{a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m}$. Thus, by Proposition A.9, $coSEMI\_HULL(a_1, b_1, \ldots, a_n, b_n) \subseteq coINTER^n_{a'_1, b'_1, \ldots, a'_n, b'_n}$ and $REV\_coSEMI\_HULL(c_1, d_1, \ldots, c_m, d_m) \subseteq REV\_coINTER^m_{c'_1, d'_1, \ldots, c'_m, d'_m}$. Now part (D) follows from part (C).

(E) Let $(x_1, y_1) \in coSEMI\_HULL^j_{a_1, b_1, \ldots, a_j, b_j}$ be such that $1 + \sum_{1 \leqslant i < j} b_i(x_1 \dot{-} a_i) \geqslant y_1 > \sum_{1 \leqslant i < j} b_i(x_1 \dot{-} a_i)$. Note that there exists such (x_1, y_1). By Proposition A.7 if $(x_1, y_1) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c''_{\min(j,m)}, d''_{\min(j,m)})$, then $(x_1, y_1) \in coINTER(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j)$. Thus, by Proposition 24, there exists a $B' \in \mathbf{rat}^+$ such that if $(x_1, y_1) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c''_{\min(j,m)}, d''_{\min(j,m)})$, then $a''_j < x_1$ and $b''_j > B'$.

Similarly, let $(x_2, y_2) \in REV\_coSEMI\_HULL^{\min(j,m)}_{c_1, d_1, \ldots, c_{\min(j,m)}, d_{\min(j,m)}}$ be such that $1 + \sum_{1 \leqslant i < \min(j,m)} b_i(y_2 \dot{-} a_i) \geqslant x_2 > \sum_{1 \leqslant i < \min(j,m)} b_i(y_2 \dot{-} a_i)$. Note that there exists such (x_2, y_2). By Proposition A.8 if $(x_2, y_2) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c''_{\min(j,m)}, d''_{\min(j,m)})$, then $(x_2, y_2) \in REV\_coINTER(c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c''_{\min(j,m)}, d''_{\min(j,m)})$. Thus, by Proposition 24, there exists a $B'' \in \mathbf{rat}^+$ such that if $(x_2, y_2) \in coINT\_OP\_HULL(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a''_j, b''_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c''_{\min(j,m)}, d''_{\min(j,m)})$, then $c''_{\min(j,m)} < y_2$ and $d''_{\min(j,m)} > B''$.

Thus, for (x_1, y_1) and (x_2, y_2) to be in $coINT\_OP\_HULL^{j,\min(j,m)}(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)})$, where $b'_j + \sum_{1 \leqslant i < j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1 \leqslant i < \min(j,m)} d_i}$, we must have $a'_j \leqslant x_1$, $b'_j \geqslant B'$, $c'_{\min(j,m)} \leqslant y_2$, $d'_{\min(j,m)} \geqslant B''$, and $b'_j \leqslant 1/B''$ and $d_{\min(j,m)} \leqslant 1/B'$. Since there are only finitely many such $(a'_j, b'_j, c'_{\min(j,m)}, d'_{\min(j,m)})$ with $b'_j, d'_{\min(j,m)} \in S$, we have part (E) using part (A).

(F) Let $X = \bigcup_{1 \leqslant j \leqslant n} X_j$ where X_j is as in part (E). Now, if $map(a'_1, b'_1, \ldots, a'_n, b'_n) <_Q map(a_1, b_1, \ldots, a_n, b_n)$, there exists a j, $1 \leqslant j \leqslant n$ such for $1 \leqslant i < j$, $a_i = a'_i$ and $b_i = b'_i$ and $Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) > Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$.

Part (F) now follows from part (E), and definition of X. $\square$

We now continue with the proof of the theorem. The aim is to construct Θ which maps $coOP\_HULL^{n,m,S}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ to $L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$.

Note that definition of Ψ mapping grammar sequence converging to a grammar for $L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$ to a grammar sequence converging to a grammar for $coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$ would be trivial. We thus just define Θ.

Without loss of generality, we will be giving Θ as mapping sets to sets.

For any finite $X \subseteq N^2$, let $Prop(X, a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$ be true iff the following two properties are satisfied:

(A) $(a_1, b_1, \ldots, a_n, b_n) \in VALID_S$, $(c_1, d_1, \ldots, c_n, d_n) \in VALID_S$, and $\sum_{1 \leqslant i \leqslant n} b_i < \frac{1}{\sum_{1 \leqslant i \leqslant m} d_i}$.

(B) For all j, $1 \leqslant j \leqslant n$, for all $a'_j, c'_j \in N, b'_j, d'_{\min(j,m)} \in S$, such that $b'_j + \sum_{1 \leqslant i < j} b_i < \frac{1}{d'_{\min(j,m)} + \sum_{1 \leqslant i < \min(j,m)} d_i}$.

If $Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a'_j, b'_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c'_{\min(j,m)}, d'_{\min(j,m)}) > Icode(a_1, b_1, \ldots, a_{j-1}, b_{j-1}, a_j, b_j; c_1, d_1, \ldots, c_{\min(j,m)-1}, d_{\min(j,m)-1}, c_{\min(j,m)}, d_{\min(j,m)})$, then

$$[X \nsubseteq coINT\_OP\_HULL^{j,\min(j,m)}_{a_1,b_1,\ldots,a_{j-1},b_{j-1},a'_j,b'_j;c_1,d_1,\ldots,c_{\min(j,m)-1},d_{\min(j,m)-1},c'_{\min(j,m)},d'_{\min(j,m)}}].$$

Note that whether $X, a_1, b_1, \ldots, a_n, b_n, c_1, d_1, \ldots, c_m, d_m$, satisfy (A) and (B), for all j, $1 \leqslant j \leqslant n$, is effectively testable.

Moreover, (B) above implies that

(C) for all $(a'_1, b'_1, \ldots, a'_n, b'_n), (c'_1, d'_1, \ldots, c'_m, d'_m) \in VALID_S$, $\sum_{1 \leqslant i \leqslant n} b'_i < \frac{1}{1 \leqslant i \leqslant m} d'_i$, such that $map(a'_1, b'_1, \ldots, a'_n, b'_n; c'_1, d'_1, \ldots, c'_m, d'_m) <_Q map(a_1, b_1, \ldots, a_n, b_n; c_1, d_1, \ldots, c_m, d_m)$, $X \nsubseteq coOP\_HULL^{n,m}_{a'_1,b'_1,\ldots,a'_n,b'_n;c'_1,d'_1,\ldots,c'_m,d'_m}$.

Thus, for finite $X \subseteq N^2$, let

$$\Theta(X) = \bigcup \{L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n)} \mid Prop(X, a_1, b_1, \ldots, a_n, b_n)\}.$$

For infinite X', $\Theta(X') = \bigcup_{X \subseteq X',\, card(X) < \infty} \Theta(X)$.

It is easy to verify that

(1) for any $X \subseteq coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$,

$$\Theta(X) \subseteq L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$$

(by (C) and the fact that for any valid I and I', $map(I) <_Q map(I')$, implies $L^{Q}_{map(I)} \subseteq L^{Q}_{map(I')}$), and

(2) for any S-valid $(a_1, b_1, \ldots, a_n, b_n)$ and $(c_1, d_1, \ldots, c_m, d_m)$, by Claim A.6(F) there exists a finite set $X \subseteq coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}$, such that $\Theta(X) \supseteq L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}$.

Thus, we have that

$$\Theta(coOP\_HULL^{n,m}_{a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m}) = L^{Q}_{map(a_1,b_1,\ldots,a_n,b_n;c_1,d_1,\ldots,c_m,d_m)}. \qquad \square$$

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